# Algebraicity modulo $\mathbf{p}$ of generalized hypergeometric series ${ }_{n} F_{n-1}$ 

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#### Abstract

Let $f(z)={ }_{n} F_{n-1}(\boldsymbol{\alpha}, \boldsymbol{\beta})$ be the hypergeometric series with parameters $\boldsymbol{\alpha}=$ $\left(\alpha_{1}, \ldots, \alpha_{n}\right)$ and $\boldsymbol{\beta}=\left(\beta_{1}, \ldots, \beta_{n-1}, 1\right)$ in $(\mathbb{Q} \cap(0,1])^{n}$, let $d_{\boldsymbol{\alpha}, \boldsymbol{\beta}}$ be the least common multiple of the denominators of $\alpha_{1}, \ldots, \alpha_{n}, \beta_{1}, \ldots, \beta_{n-1}$ written in lowest form and let $p$ be a prime number such that $p$ does not divide $d_{\boldsymbol{\alpha}, \boldsymbol{\beta}}$ and $f(z) \in \mathbb{Z}_{(p)}[[z]]$. Recently in [11], it was shown that if for all $i, j \in\{1, \ldots, n\}, \alpha_{i}-\beta_{j} \notin \mathbb{Z}$ then the reduction of $f(z)$ modulo $p$ is algebraic over $\mathbb{F}_{p}(z)$. A standard way to measure the complexity of an algebraic power series is to estimate its degree and its height. In this work, we prove that if $p>2 d_{\boldsymbol{\alpha}, \boldsymbol{\beta}}$ then there is a nonzero polynomial $P_{p}(Y) \in \mathbb{F}_{p}(z)[Y]$ having degree at most $p^{2^{n}} \varphi\left(d_{\boldsymbol{\alpha}, \boldsymbol{\beta}}\right)$ and height at most $5^{n}(n+1)!p^{2^{n}} \varphi\left(d_{\boldsymbol{\alpha}, \boldsymbol{\beta}}\right)$ such that $P_{p}(f(z) \bmod p)=0$, where $\varphi$ is the Euler's totient function. Furthermore, our method of proof provides us a way to make an explicit construction of the polynomial $P_{p}(Y)$. We illustrate this construction by applying it to some explicit hypergeometric series.


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## 1. Introduction

Let $\boldsymbol{\alpha}=\left(\alpha_{1}, \ldots, \alpha_{n}\right)$ and $\boldsymbol{\beta}=\left(\beta_{1}, \ldots, \beta_{n-1}, 1\right)$ be in $\left(\mathbb{Q} \backslash \mathbb{Z}_{\leqslant 0}\right)^{n}$. The generalized hypergeometric series with parameters $\boldsymbol{\alpha}, \boldsymbol{\beta}$ is the power series given by

$$
{ }_{n} F_{n-1}(\boldsymbol{\alpha}, \boldsymbol{\beta} ; z)=\sum_{i \geqslant 0} \mathcal{Q}_{\boldsymbol{\alpha}, \boldsymbol{\beta}}(i) z^{i} \in \mathbb{Q}[[z]] \text { with } \mathcal{Q}_{\boldsymbol{\alpha}, \boldsymbol{\beta}}(i)=\frac{\left(\alpha_{1}\right)_{i} \cdots\left(\alpha_{n}\right)_{i}}{\left(\beta_{1}\right)_{i} \cdots\left(\beta_{n-1}\right)_{i} i!},
$$

where for a real number $x$ and a nonnegative integer $i,(x)_{i}$ is the Pochhammer symbol, that is, $(x)_{0}=1$ and $(x)_{i}=x(x+1) \cdots(x+i-1)$ for $i>0$. We denote by $d_{\boldsymbol{\alpha}, \boldsymbol{\beta}}$ the least common multiple of the denominators of $\alpha_{1}, \ldots, \alpha_{n}$ and $\beta_{1}, \ldots, \beta_{n-1}$ written in lowest form. It is well-known that ${ }_{n} F_{n-1}(\boldsymbol{\alpha}, \boldsymbol{\beta} ; z)$ is a solution of the hypergeometric operator

$$
\mathcal{H}(\boldsymbol{\alpha}, \boldsymbol{\beta})=\prod_{i=1}^{n}\left(\delta+\beta_{i}-1\right)-z \prod_{i=1}^{n}\left(\delta+\alpha_{i}\right), \text { with } \delta=z \frac{d}{d z}
$$

We recall that for any field $K$, the power series $h(z) \in K[[z]]$ is an algebraic power series over $K(z)$ if there exists a nonzero polynomial $P(Y) \in K(z)[Y]$ such that $P(h(z))=0$. Given a prime number $p$, we denote by $\mathbb{Z}_{(p)}$ the localization of $\mathbb{Z}$ at ideal $(p)$. That is, $\mathbb{Z}_{(p)}$ is the set of rational numbers $a / b$ written in lowest form such that $p$ does not divide $b$. This ring is a local ring whose maximal ideal is $(p) \mathbb{Z}_{(p)}$ and its residue field is the field with $p$ elements, which is denoted by $\mathbb{F}_{p}$. Given a power series $f(z)=\sum_{i \geqslant 0} a(i) \in \mathbb{Z}_{(p)}[[z]]$, the reduction of $f$ modulo $p$ is $f(z) \bmod p:=\sum_{i \geqslant 0}(a(i) \bmod p) z^{i} \in \mathbb{F}_{p}[[z]]$. The power series $f(z)$ is said to be algebraic modulo $p$ if $f(z) \bmod p$ is an algebraic power series over $\mathbb{F}_{p}(z)$. A usual way to measure the complexity of an algebraic power series is to estimate its degree and its height.

Definition 1.1. - Let $K$ be a field and let $a(z)=s(z) / t(z)$ be in $K(z)$ written in lowest form. The height of $a(z)$ is equal to $\max \{\operatorname{deg}(s(z)), \operatorname{deg}(t(z))\}$. Let $P(Y)=\sum_{i=0}^{m} a_{i}(z) Y^{i}$ be in $K(z)[Y]$ such that $a_{m}(z)$ is not zero. The degree of $P$ is $m$ and the height of $P$ is the maximum of the heights of $a_{0}(z), \ldots, a_{m}(z)$.

We have shown in [11, Theorem 1.2] the following result. Let $\mathcal{S}$ be an infinite set of prime numbers $p$ such that $p$ does not divide $d_{\boldsymbol{\alpha}, \boldsymbol{\beta}}$ and ${ }_{n} F_{n-1}(\boldsymbol{\alpha}, \boldsymbol{\beta} ; z) \in \mathbb{Z}_{(p)}[[z]]$. We proved that if, for all $i, j \in\{1, \ldots, n\}, \alpha_{i}-\beta_{j} \notin \mathbb{Z}$ then, for all $p \in \mathcal{S},{ }_{n} F_{n-1}(\boldsymbol{\alpha}, \boldsymbol{\beta} ; z)$ is algebraic modulo $p$. We also established that ${ }_{n} F_{n-1}(\boldsymbol{\alpha}, \boldsymbol{\beta} ; z) \bmod p$ has degree at most $p^{n^{2} \varphi\left(d_{\boldsymbol{\alpha}, \boldsymbol{\beta}}\right)}$, where $\varphi$ is the Euler's totient function. However, the result obtained in [11] does not offer any information about the height. The main result of this work shows that if $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ belong to $(\mathbb{Q} \cap(0,1])^{n}$ then, for all $p \in \mathcal{S}$ satisfying $p>2 d_{\boldsymbol{\alpha}, \boldsymbol{\beta}}$, there is a nonzero polynomial $P_{p}(Y) \in \mathbb{F}_{p}(z)[Y]$ having degree at most $p^{2^{t} l}$ and height at most $5^{t}(t+1)!p^{2^{t} l}$ such that $P_{p}\left({ }_{n} F_{n-1}(\boldsymbol{\alpha}, \boldsymbol{\beta} ; z) \bmod p\right)=0$, where $t \leqslant n$ and $l$ is the order of $p$ in $\left(\mathbb{Z} / d_{\boldsymbol{\alpha}, \boldsymbol{\beta}} \mathbb{Z}\right)^{*}$. Further, the advantage of the present method is that it gives an explicit way to construct the polynomial $P_{p}(Y)$.

## ALGEBRAICITY MODULO P OF GENERALIZED HYPERGEOMETRIC SERIES ${ }_{n} F_{n-1}$

### 1.1. Main result

In order to state our main result, Theorem 1.3, we have to introduce some notations. Let $p$ be a prime number such that $p$ does not divide $d_{\boldsymbol{\alpha}, \boldsymbol{\beta}}$. Then $\mathcal{H}(\boldsymbol{\alpha}, \boldsymbol{\beta}) \in \mathbb{Z}_{(p)}[z][\delta]$. In particular, we can reduce $\mathcal{H}(\boldsymbol{\alpha}, \boldsymbol{\beta})$ modulo $p$ and we denote by $\mathcal{H}(\boldsymbol{\alpha}, \boldsymbol{\beta}, p)$ its reduction modulo $p$. That is,

$$
\mathcal{H}(\boldsymbol{\alpha}, \boldsymbol{\beta}, p):=\prod_{i=1}^{n}\left(\delta+\left(\beta_{i}-1\right) \bmod p\right)-z \prod_{i=1}^{n}\left(\delta+\alpha_{i} \bmod p\right) \in \mathbb{F}_{p}[z][\delta]
$$

An element of the set $\left\{0,1-\beta_{1} \bmod p, \ldots, 1-\beta_{n-1} \bmod p\right\}$ will be called an exponent at zero of $\mathcal{H}(\boldsymbol{\alpha}, \boldsymbol{\beta}, p)$. Consider the following set:

$$
E_{\boldsymbol{\alpha}, \boldsymbol{\beta}, p}=\{r \in\{0,1, \ldots, p-1\}: r \bmod p \text { is an exponent at zero of } \mathcal{H}(\boldsymbol{\alpha}, \boldsymbol{\beta}, p)\} .
$$

Given a finite set $E$, by $\# E$ we mean the number of elements of $E$. It is clear that $0 \in E_{\boldsymbol{\alpha}, \boldsymbol{\beta}, p}$ and that $\# E_{\boldsymbol{\alpha}, \boldsymbol{\beta}, p} \leqslant n$. As usual, $v_{p}: \mathbb{Q} \rightarrow \mathbb{Z}$ denotes the $p$-adic valuation map. We define the following set:

$$
S_{\boldsymbol{\alpha}, \boldsymbol{\beta}, p}=\left\{r \in\{0,1, \ldots, p-1\}: r \in E_{\boldsymbol{\alpha}, \boldsymbol{\beta}, p} \text { and } v_{p}\left(\mathcal{Q}_{\boldsymbol{\alpha}, \boldsymbol{\beta}}(r)\right)=0\right\}
$$

The set $S_{\boldsymbol{\alpha}, \boldsymbol{\beta}, p}$ is not empty because $0 \in S_{\boldsymbol{\alpha}, \boldsymbol{\beta}, p}$, and $\# S_{\boldsymbol{\alpha}, \boldsymbol{\beta}, p} \leqslant n$ since $S_{\boldsymbol{\alpha}, \boldsymbol{\beta}, p} \subset E_{\boldsymbol{\alpha}, \boldsymbol{\beta}, p}$.
Let us recall the definition of the map $\mathfrak{D}_{p}: \mathbb{Z}_{(p)} \rightarrow \mathbb{Z}_{(p)}$ introduced by Dwork in $[9$, Chap. 8]. The map $\mathfrak{D}_{p}: \mathbb{Z}_{(p)} \rightarrow \mathbb{Z}_{(p)}$ is such that, for every $\gamma$ in $\mathbb{Z}_{(p)}, \mathfrak{D}_{p}(\gamma)$ is the unique element in $\mathbb{Z}_{(p)}$ such that $p \mathfrak{D}_{p}(\gamma)-\gamma$ belongs to $\{0, \ldots, p-1\}$. In [9, Chap. 8] this map is denoted by $\gamma \mapsto \gamma^{\prime}$. For $\gamma=\left(\gamma_{1}, \ldots, \gamma_{n}\right) \in \mathbb{Z}_{(p)}^{n}$ we write $\mathfrak{D}_{p}(\gamma)$ for $\left(\mathfrak{D}_{p}\left(\gamma_{1}\right), \ldots, \mathfrak{D}_{p}\left(\gamma_{n}\right)\right)$. For all integers $m \geqslant 1, \mathfrak{D}_{p}^{m}$ is the $m$-th composition of $\mathfrak{D}_{p}$ with itself and $\mathfrak{D}_{p}^{0}$ is identity map on $\mathbb{Z}_{(p)}$.

Remark 1.2.- Let $\boldsymbol{\alpha}=\left(\alpha_{1}, \ldots, \alpha_{n}\right)$ and $\boldsymbol{\beta}=\left(\beta_{1}, \ldots, \beta_{n-1}, 1\right)$ be in $\mathbb{Q}^{n}$ and let $p$ a prime number such that $p$ does not divide $d_{\boldsymbol{\alpha}, \boldsymbol{\beta}}$. Then, $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ belong to $\mathbb{Z}_{(p)}^{n}$ and for this reason, for all integers $m \geqslant 0$, the differential operator $\mathcal{H}\left(\mathfrak{D}_{p}^{m}(\boldsymbol{\alpha}), \mathfrak{D}_{p}^{m}(\boldsymbol{\beta})\right)$ belongs to $\mathbb{Z}_{(p)}[z][\delta]$. Thus, for all integers $m \geqslant 0$, the sets $E_{\mathfrak{D}_{p}^{m}(\boldsymbol{\alpha}), \mathfrak{D}_{p}^{m}(\boldsymbol{\beta}), p}, S_{\mathfrak{D}_{p}^{m}(\boldsymbol{\alpha}), \mathfrak{D}_{p}^{m}(\boldsymbol{\beta}), p}$ are well-defined.

We are now ready to state our main result:
Theorem 1.3.-Let $\boldsymbol{\alpha}=\left(\alpha_{1}, \ldots, \alpha_{n}\right)$ and $\boldsymbol{\beta}=\left(\beta_{1}, \ldots, \beta_{n-1}, 1\right)$ be in $(\mathbb{Q} \cap(0,1])^{n}$, let $f(z)$ be the hypergeometric series ${ }_{n} F_{n-1}(\boldsymbol{\alpha}, \boldsymbol{\beta} ; z)$, let $p$ be prime number such that $p>2 d_{\boldsymbol{\alpha}, \boldsymbol{\beta}}$ and $f(z) \in \mathbb{Z}_{(p)}[[z]]$, and let $\left(\mathbb{Z} / d_{\boldsymbol{\alpha}, \boldsymbol{\beta}} \mathbb{Z}\right)^{*}$ be the unit group of $\mathbb{Z} / d_{\boldsymbol{\alpha}, \boldsymbol{\beta}} \mathbb{Z}$. Suppose that, for all $i, j \in\{1, \ldots, n\}, \alpha_{i}-\beta_{j} \notin \mathbb{Z}$. Then there is a nonzero polynomial $P_{p}(Y) \in \mathbb{F}_{p}(z)[Y]$ having degree at most $p^{2^{n} \varphi\left(d_{\boldsymbol{\alpha}, \boldsymbol{\beta}}\right)}$ and height at most $5^{n}(n+1)!p^{2^{n} \varphi\left(d_{\alpha, \boldsymbol{\beta}}\right)}$ such that $P_{p}(f(z) \bmod p)=$ 0 . Moreover, if $l$ is the order of $p$ in $\left(\mathbb{Z} / d_{\boldsymbol{\alpha}, \boldsymbol{\beta}} \mathbb{Z}\right)^{*}$ then the following assertions hold:

$$
\begin{aligned}
& \text { (1) if } 1=\# S_{\mathfrak{D}_{p}^{l-1}(\boldsymbol{\alpha}), \mathfrak{D}_{p}^{l-1}(\boldsymbol{\beta}), p} \text { then } \\
& \qquad P_{p}(Y)=Y-Q_{1}(z) Y^{p^{l}}
\end{aligned}
$$

where $Q_{1}(z)$ belongs to $\mathbb{F}_{p}[z]$ and has degree less than $p^{l}$;
(2)

$$
\begin{aligned}
& \text { if } 2=\# S_{\mathfrak{D}_{p}^{l-1}(\boldsymbol{\alpha}), \mathfrak{D}_{p}^{l-1}(\boldsymbol{\beta}), p} \text { then } \\
& \qquad P_{p}(Y)=Y-Q_{1}(z) Y^{p^{l}}-Q_{2}(z) Y^{p^{2 l}},
\end{aligned}
$$

and the height of $P_{p}(Y)$ is less than $2 p^{2 l}$;
(3) if $2<\# S_{\mathfrak{D}_{p}^{l-1}(\boldsymbol{\alpha}), \mathfrak{D}_{p}^{l-1}(\boldsymbol{\beta}), p}=t+1$ then

$$
P_{p}(Y)=Y-Q_{1}(z) Y^{p^{i}}-Q_{2}(z) Y^{p^{2 l}}-Q_{3}(z) Y^{p^{3 l}}-\cdots-Q_{2^{t}}(z) Y^{p^{2^{t} l}},
$$

and the height of $P_{p}(Y)$ is less than $5^{t}(t+1)!p^{2^{t} l}$.
Let us make a few comments. In these comments we keep the notations used in the statement of Theorem 1.3.

- To prove Theorem 1.3 it is sufficient to show that the assertions (1), (2) and (3) hold because $l \leqslant \varphi\left(d_{\boldsymbol{\alpha}, \boldsymbol{\beta}}\right)=\#\left(\mathbb{Z} / d_{\boldsymbol{\alpha}, \boldsymbol{\beta}} \mathbb{Z}\right)^{*}$ and $\# S_{\mathfrak{D}_{p}^{l-1}(\boldsymbol{\alpha}), \mathfrak{D}_{p}^{l-1}(\boldsymbol{\beta}), p} \leqslant n$.
- The method of proof of Theorem 1.3 provides us a way to make an explicit construction of the polynomial $P_{p}(Y) \in \mathbb{F}_{p}(z)[Y]$. In Section 9 , we show how to construct the polynomial $P_{p}(Y)$ and we illustrate this construction by applying it to some hypergeometric series.
- The conclusion of the assertion (1) of Theorem 1.3 is to equivalent to saying that the hypergeometric series ${ }_{n} F_{n-1}(\boldsymbol{\alpha}, \boldsymbol{\beta}, z)$ satisfies the $p^{l}$-Lucas property. We say that a power series $f(z)=\sum_{i \geqslant 0} a(i) z^{i} \in \mathbb{Q}[[z]]$ satisfies the $p^{l}$-Lucas property if $f(z) \in \mathbb{Z}_{(p)}[[z]], a(0)=1$ and, for all integers $m \geqslant 0$ and for all $r \in\left\{0, \ldots, p^{l}-1\right\}, a\left(m p^{l}+r\right) \equiv a(m) a(r) \bmod p$. From [2, Proposition 4.8], it follows that $f(z) \in 1+z \mathbb{Z}_{(p)}[[z]]$ satisfies the $p^{l}$-Lucas property if and only if $f \equiv A_{p}(z) f^{p^{l}} \bmod p$, where $A_{p}$ is a polynomial with coefficients in $\mathbb{Z}_{(p)}$ having degree less than $p^{l}$.
- As we have already said, from Theorem 1.2 of [11] it follows that ${ }_{n} F_{n-1}(\boldsymbol{\alpha}, \boldsymbol{\beta} ; z)$ is algebraic modulo $p$ and the degree of its reduction modulo $p$ is at most $p^{n^{2} \varphi\left(d_{\boldsymbol{\alpha}, \boldsymbol{\beta}}\right)}$. The proof of this result relies on the fact that $\mathcal{H}(\boldsymbol{\alpha}, \boldsymbol{\beta})$ has a strong Frobenius structure for all $p \in \mathcal{S}$ with period $\varphi\left(d_{\boldsymbol{\alpha}, \boldsymbol{\beta}}\right)$. Nevertheless, the approach used in this work to prove Theorem 1.3 does not use the existence of strong Frobenius structure. ${ }^{(1)}$
- In Section 2 we will compare through some hypergeometric series the estimate $p^{2^{t} l}$ given by Theorem 1.3 and the estimate $p^{n^{2} \varphi\left(d_{\alpha, \boldsymbol{\beta}}\right)}$ given by Theorem 1.2 of [11]. As we will see, for these particular examples, the estimate $p^{2^{t} l}$ is much finer than the estimate $p^{n^{2} \varphi\left(d_{\alpha, \boldsymbol{\beta}}\right)}$.


### 1.2. Structure of proof

The proof of Theorem 1.3 is based on Theorem 3.2. The latter one is derived from Propositions 4.1 and 4.2. In section 5, Proposition 4.2 is proved. Proposition 4.1 will be

[^1]proved in Section 6 and its proof relies on Lemmas 6.1 and 6.2. The proof of Lemma 6.1 is given in Section 7. Finally, in Section 8 we prove Lemma 6.2. Nevertheless, the proof of this lemma depends on Lemma 8.1, which is also proved in Section 8. Lemma 8.1 is, in fact, the main ingredient of this work and its proof is based essentially on two facts. The first one deals with some $p$-adic properties of the sequence $\left\{\mathcal{Q}_{\boldsymbol{\alpha}, \boldsymbol{\beta}}(j)\right\}_{j \geqslant 0}$. Sections 8.2 and 8.4 are devoted to studying these $p$-adic properties. The second fact is the equality
\[

$$
\begin{equation*}
I(j) \mathcal{Q}_{\boldsymbol{\alpha}, \boldsymbol{\beta}}(j)=\mathcal{Q}_{\boldsymbol{\alpha}, \boldsymbol{\beta}}(j-1) T(j-1) \tag{1.1}
\end{equation*}
$$

\]

for all integers $j \geqslant 1$, where $I(j)=\prod_{i=1}^{n}\left(j+\beta_{i}-1\right)$ and $T(j)=\prod_{i=1}^{n}\left(j+\alpha_{i}\right)$. The Equality (1.1) is equivalent to the fact that ${ }_{n} F_{n-1}(\boldsymbol{\alpha}, \boldsymbol{\beta} ; z)$ is solution of $\mathcal{H}(\boldsymbol{\alpha}, \boldsymbol{\beta})$.

### 1.3. Reduction modulo $p$ of generalized hypergeometric series

In oder to apply Theorem 1.3, a natural question is to determine when it is possible to reduce a hypergeometric series modulo $p$. In this direction, an interesting class of hypergeometric series is the class of globally bounded hypergeometric series. We say that the hypergeometric series ${ }_{n} F_{n-1}(\boldsymbol{\alpha}, \boldsymbol{\beta} ; z)$ is globally bounded if there is $c \in \mathbb{Q} \backslash\{0\}$ such that ${ }_{n} F_{n-1}(\boldsymbol{\alpha}, \boldsymbol{\beta} ; c z)$ belongs to $\mathbb{Z}[[z]]$. Consequently, a globally bounded hypergeometric series can be reduced modulo $p$ for almost every prime number $p$. As an example, the hypergeometric series $\mathfrak{g}(z):={ }_{3} F_{2}(\boldsymbol{\alpha}, \boldsymbol{\beta} ; z)$ with parameters $\boldsymbol{\alpha}=\left(\frac{1}{9}, \frac{4}{9}, \frac{5}{9}\right)$ and $\boldsymbol{\beta}=\left(\frac{1}{3}, 1,1\right)$ is globally bounded because $\mathfrak{g}\left(27^{2} z\right) \in \mathbb{Z}[[z]]$. In [6], Christol has given a characterization of the hypergeometric series that are globally bounded. For more exemples of globally bounded hypergeometric series we refer the reader to $[1,4]$.

In addition, there are also many generalized hypergeometric series that are not globally bounded but, for infinitely many prime numbers $p$, they can be reduced modulo $p$. For example, $\mathfrak{f}(z)={ }_{2} F_{1}(\boldsymbol{\alpha}, \boldsymbol{\beta} ; z)$ with $\boldsymbol{\alpha}=\left(\frac{1}{3}, \frac{1}{2}\right)$ and $\boldsymbol{\beta}=\left(\frac{5}{12}, 1\right)$ is not globally bounded but thanks to Proposition 24 of [8], for all primes $p \equiv 1 \bmod 12, \mathfrak{f}(z) \in \mathbb{Z}_{(p)}[[z]]$.

## 2. Examples

The aim of this section is to illustrate Theorem 1.3 by applying it to the hypergeometric series $\mathfrak{f}(z)$ and $\mathfrak{g}(z)$. In order to proceed, we need some results which will also be useful in the rest of the paper.

Lemma 2.1. - Let $\gamma=\frac{a}{b}$ be in $\mathbb{Q} \cap(0,1]$ written in lowest form and let $p$ be a prime number such $v_{p}(\gamma)=0$. If $p^{l} \equiv 1 \bmod b$ then $\mathfrak{D}_{p}^{l}(\gamma)=\gamma$.

Proof. - Since $\gamma \in \mathbb{Z}_{(p)}$, we have

$$
\gamma=\sum_{s \geqslant 0} j_{s} p^{s}
$$

where, for all $s, j_{s} \in\{0, \ldots, p-1\}$. Note that $j_{0} \neq 0$ because $v_{p}(\gamma)=0$. First, we are going to show by induction on $n \in \mathbb{N}_{>0}$ that

$$
\mathfrak{D}_{p}^{n}(\gamma)=1+\sum_{s \geqslant n} j_{s} p^{s-n}
$$

It is clear that, $p\left(1+\sum_{s \geqslant 1} j_{s} p^{s-1}\right)-\gamma=p-j_{0}$. Then, $\mathfrak{D}_{p}(\gamma)=1+\sum_{s \geqslant 1} j_{s} p^{s-1}$ because $p-j_{0} \in\{1, \ldots, p-1\}$. Now, suppose that $\mathfrak{D}_{p}^{n}(\gamma)=1+\sum_{s \geqslant n} j_{s} p^{s-n}$. It is clear that

$$
p\left(1+\sum_{s \geqslant n+1} j_{s} p^{s-n-1}\right)-\left(1+\sum_{s \geqslant n} j_{s} p^{s-n}\right)=p-j_{n}-1
$$

As $p-j_{n}-1 \in\{0, \ldots, p-1\}$ and, by induction hypothesis, $\mathfrak{D}_{p}^{n}(\gamma)=1+\sum_{s \geqslant n} j_{s} p^{s-n}$ then $\mathfrak{D}_{p}^{n+1}(\gamma)=1+\sum_{s \geqslant n+1} j_{s} p^{s-n-1}$.

Thus, for all integers $n \geqslant 1$,

$$
\mathfrak{D}_{p}^{n}(\gamma)=1+\sum_{s \geqslant n} j_{s} p^{s-n}
$$

In particular for the integer $l$, we have

$$
\begin{aligned}
p^{l} \mathfrak{D}_{p}^{l}(\gamma)-\gamma & =p^{l}\left(1+\sum_{s \geqslant l} j_{s} p^{s-l}\right)-\sum_{s \geqslant 0} j_{s} p^{s} \\
& =p^{l}-\sum_{s=0}^{l-1} j_{s} p^{s}
\end{aligned}
$$

As $j_{0} \in\{1, \ldots, p-1\}$ and for all $s \geqslant 1, j_{s} \in\{0, \ldots, p-1\}$, then $p^{l}-\sum_{s=0}^{l-1} j_{s} p^{s} \in$ $\left\{1, \ldots, p^{l}-1\right\}$. Hence, $p^{l} \mathfrak{D}_{p}^{l}(\gamma)-\gamma \in\left\{1, \ldots, p^{l}-1\right\}$.

We now prove that $p^{l} \gamma-\gamma \in\left\{1, \ldots, p^{l}-1\right\}$. Indeed, as $p^{l} \equiv 1 \bmod b$ then $p^{l}=1+b k$. So, $p^{l} \gamma=\gamma+a k$. We also have $a \leqslant b$ because by assumption $\gamma \in(0,1]$. Thus, $a k \leqslant b k$ and as $b k=p^{l}-1$ then $a k \leqslant p^{l}-1$. Therefore, $p^{l} \gamma-\gamma \in\left\{1, \ldots, p^{l}-1\right\}$.

So that $p^{l} \mathfrak{D}_{p}^{l}(\gamma)-\gamma$ and $p^{l} \gamma-\gamma$ belong to $\left\{1, \ldots, p^{l}-1\right\}$. Without losing any generality we can assume that $p^{l} \mathfrak{D}_{p}^{l}(\gamma)-\gamma \geqslant p^{l} \gamma-\gamma$. Then, $p^{l} \mathfrak{D}_{p}^{l}(\gamma)-p^{l} \gamma=p^{l}\left(\mathfrak{D}_{p}^{l}(\gamma)-\gamma\right)$ belongs to $\left\{0,1 \ldots, p^{l}-1\right\}$. We write $\mathfrak{D}_{p}^{l}(\gamma)-\gamma=\frac{c}{d} \in \mathbb{Z}_{(p)}$ where $c$ and $d$ are positive co-prime integers. Hence, $p^{l} c=d t$ with $t \in\left\{0,1 \ldots, p^{l}-1\right\}$. We assume for contradiction that $\frac{c}{d} \neq 0$. As $p$ does not divide $d$, then $p^{l}$ divides $t$. This is a clear contradiction of the fact that $t$ belongs to $\left\{0,1 \ldots, p^{l}-1\right\}$. Consequently, $\frac{c}{d}=0$, that is, $\mathfrak{D}_{p}^{l}(\gamma)-\gamma=0$.

Lemma 2.2. - Let $p$ be a prime number and $\gamma$ be in $\mathbb{Z}_{(p)}$. We put $s:=p \mathfrak{D}_{p}(\gamma)-\gamma$. If $v_{p}\left(\mathfrak{D}_{p}(\gamma)\right)=0$ then, for every $r \in\{0,1, \ldots, p-1\}$, we have

$$
v_{p}\left((\gamma)_{r}\right)=\left\{\begin{array}{ccc}
0 & \text { if } & r \leqslant s \\
1 & \text { if } & r>s
\end{array}\right.
$$

Proof. - By definition of the map $\mathfrak{D}_{p}, s$ is the unique integer in $\{0, \ldots, p-1\}$ such that $\gamma+s \in p \mathbb{Z}_{(p)}$. For this reason, $v_{p}((\gamma) \cdots(\gamma+s-1)(\gamma+s+1) \cdots(\gamma+p-1))=0$. So, if $r \leqslant s, v_{p}\left((\gamma)_{r}\right)=0$ and if $r>s, v_{p}\left((\gamma)_{r}\right)=v_{p}\left(p \mathfrak{D}_{p}(\gamma)\right)=1+v_{p}\left(\mathfrak{D}_{p}(\gamma)\right)=1$ because by assumption, $v_{p}\left(\mathfrak{D}_{p}(\gamma)\right)=0$.

Example 2.3. - Consider the hypergeometric series $\mathfrak{f}(z):={ }_{2} F_{1}(\boldsymbol{\alpha}, \boldsymbol{\beta} ; z)$, with $\boldsymbol{\alpha}=$ $\left(\frac{1}{3}, \frac{1}{2}\right)$ and $\boldsymbol{\beta}=\left(\frac{5}{12}, 1\right)$. In this case $d_{\boldsymbol{\alpha}, \boldsymbol{\beta}}=12$. Let $\mathcal{S}$ be the set of prime numbers $p$ such that $p>24$ and $p \equiv 1 \bmod 12$. So, by applying Proposition 24 of $[8]$, we conclude that, for every $p \in \mathcal{S}, \mathfrak{f}(z) \in \mathbb{Z}_{(p)}[[z]]$. From Theorem 1.2 of [11], we get that, for every $p \in \mathcal{S}, \mathfrak{f}(z) \bmod p$ has degree at most $p^{16}$. Actually, we will see that, by applying Theorem 1.3, $\mathfrak{f}(z) \bmod p$ has degree at most $p^{2}$ for all $p \in \mathcal{S}$. Let $p$ be in $\mathcal{S}$. Then, $p=1+12 k$ with $k>1$. We first prove that $S_{\boldsymbol{\alpha}, \boldsymbol{\beta}, p}=\{0,1+5 k\}$. It is nor hard to see that $E_{\boldsymbol{\alpha}, \boldsymbol{\beta}, p}=\{0,1+5 k\}$ and it is clear that $0 \in S_{\boldsymbol{\alpha}, \boldsymbol{\beta}, p}$. As $p \equiv 1 \bmod 12$ and $v_{p}(\boldsymbol{\alpha})=(0,0)=v_{p}(\boldsymbol{\beta})$ then, from Lemma 2.1, we obtain $\mathfrak{D}_{p}(\boldsymbol{\alpha})=\boldsymbol{\alpha}$ and $\mathfrak{D}_{p}(\boldsymbol{\beta})=\boldsymbol{\beta}$. Thus, we obtain the following equalities:

$$
4 k=p \mathfrak{D}_{p}(1 / 3)-1 / 3, \quad 6 k=p \mathfrak{D}_{p}(1 / 2)-1 / 2, \text { and } 5 k=p \mathfrak{D}_{p}(5 / 12)-5 / 12
$$

So, from Lemma 2.2, we obtain

$$
v_{p}\left((1 / 3)_{1+5 k}\right)=1, \quad v_{p}\left((1 / 2)_{1+5 k}\right)=0, \text { and } v_{p}\left((5 / 12)_{1+5 k}\right)=1
$$

It is clear that $v_{p}\left((1)_{1+5 k}\right)=0$. Therefore,

$$
v_{p}\left(\frac{(1 / 3)_{1+5 k}(1 / 2)_{1+5 k}}{(5 / 12)_{1+5 k}(1)_{1+5 k}}\right)=0
$$

Whence, $1+5 k \in S_{\boldsymbol{\alpha}, \boldsymbol{\beta}, p}$. Consequently, $\# S_{\boldsymbol{\alpha}, \boldsymbol{\beta}, p}=2$. Then, it follows from (2) of Theorem 1.3 that there are $Q_{1, p}(z), Q_{2, p}(z) \in \mathbb{Q}(z) \cap \mathbb{Z}_{(p)}[[z]]\left[z^{-1}\right]$ such that

$$
\begin{equation*}
\mathfrak{f} \equiv Q_{1, p}(z) \mathfrak{f}^{p}+Q_{2, p}(z) \mathfrak{f}^{p^{2}} \bmod p \tag{2.1}
\end{equation*}
$$

and the heights of $Q_{1, p}(z) \bmod p$ and $Q_{2, p}(z) \bmod p$ are less than $2 p^{2}$.
Example 2.4. - Consider the hypergeometric series $\mathfrak{g}(z):={ }_{3} F_{2}(\boldsymbol{\alpha}, \boldsymbol{\beta} ; z)$, with $\boldsymbol{\alpha}=$ $\left(\frac{1}{9}, \frac{4}{9}, \frac{5}{9}\right)$ and $\boldsymbol{\beta}=\left(\frac{1}{3}, 1,1\right)$. In this case $d_{\boldsymbol{\alpha}, \boldsymbol{\beta}}=9$. It turns out that $\mathfrak{g}\left(27^{2} z\right) \in \mathbb{Z}[[z]]$. So that, for every prime number $p \neq 3, \mathfrak{g}(z)$ belongs to $\mathbb{Z}_{(p)}[[z]]$. From Theorem 1.2 of [11], we get that, for all primes $p \neq 3, \mathfrak{g}(z) \bmod p$ has degree at most $p^{54}$. Nevertheless, by applying Theorem 1.3, we obtain for some prime numbers $p$ a finer estimate than $p^{54}$. Let $\mathcal{S}$ be the set of prime numbers $p$ such that $p>18$ and $p \equiv 8 \bmod 9$. Then, for every $p \in \mathcal{S}, p^{2} \equiv 1 \bmod 9$. We are going to see that, for every $p \in \mathcal{S}, \# S_{\mathfrak{D}_{p}(\boldsymbol{\alpha}), \mathfrak{D}_{p}(\boldsymbol{\beta}), p}=2$. Let $p$ be in $\mathcal{S}$. We put $\boldsymbol{\alpha}^{\prime}=(8 / 9,5 / 9,4 / 9)$ and $\boldsymbol{\beta}^{\prime}=(2 / 3,1,1)$. As $p \equiv 8 \bmod 9$ then $p=8+9 k$ with $k>1$ and we also have the following equalities:

$$
\begin{array}{ll}
7+8 k=p(8 / 9)-1 / 9 & 4+5 k=p(5 / 9)-4 / 9 \\
3+4 k=p(4 / 9)-5 / 9 & 5+6 k=p(2 / 3)-1 / 3
\end{array}
$$

So that, $\mathfrak{D}_{p}(\boldsymbol{\alpha})=\boldsymbol{\alpha}^{\prime}$ and $\mathfrak{D}_{p}(\boldsymbol{\beta})=\boldsymbol{\beta}^{\prime}$. Thus, $E_{\mathfrak{D}_{p}(\boldsymbol{\alpha}), \mathfrak{D}_{p}(\boldsymbol{\beta}), p}=\left\{0,3+3 k_{p}\right\}$. Furthermore, since $p^{2} \equiv 1 \bmod 9$ and $v_{p}(\boldsymbol{\alpha})=(0,0,0)=v_{p}(\boldsymbol{\beta})$, by Lemma 2.1, we obtain $\mathfrak{D}_{p}^{2}(\boldsymbol{\alpha})=\boldsymbol{\alpha}$
and $\mathfrak{D}_{p}^{2}(\boldsymbol{\beta})=\boldsymbol{\beta}$. Therefore, $\mathfrak{D}_{p}\left(\boldsymbol{\alpha}^{\prime}\right)=\boldsymbol{\alpha}$ and $\mathfrak{D}_{p}\left(\boldsymbol{\beta}^{\prime}\right)=\boldsymbol{\beta}$ and consequently, we obtain the following equalities:

$$
\left.\begin{array}{rlrl}
k & =p(1 / 9)-(8 / 9) & & 3+4 k
\end{array}\right) p(4 / 9)-(5 / 9) .
$$

So, it follows from Lemma 2.2 that

$$
v_{p}\left((8 / 9)_{3+3 k}\right)=1, \quad v_{p}\left((5 / 9)_{3+3 k}\right)=0, \quad v_{p}\left((4 / 9)_{3+3 k}\right)=0, \text { and } v_{p}\left((2 / 3)_{3+3 k}\right)=1
$$

It is clear that $v_{p}\left((1)_{3+3 k}\right)=0$. Therefore,

$$
v_{p}\left(\frac{(8 / 9)_{3+3 k}(5 / 9)_{3+3 k}(4 / 9)_{3+3 k}}{(2 / 3)_{3+3 k}(1)_{3+3 k}^{2}}\right)=0
$$

Whence, $3+3 k_{p} \in S_{\mathfrak{D}_{p}(\boldsymbol{\alpha}), \mathfrak{D}_{p}(\boldsymbol{\beta}), p}$. And, it is clear that $0 \in S_{\mathfrak{D}_{p}(\boldsymbol{\alpha}), \mathfrak{D}_{p}(\boldsymbol{\beta}), p}$.
Consequently, $\# S_{\mathfrak{D}_{p}(\boldsymbol{\alpha}), \mathfrak{D}_{p}(\boldsymbol{\beta}), p}=2$. Then, it follows from (2) of Theorem 1.3 that, for every prime $p \in \mathcal{S}$, there are $A_{1, p}(z), A_{2, p}(z) \in \mathbb{Q}(z) \cap \mathbb{Z}_{(p)}[[z]]\left[z^{-1}\right]$ such that

$$
\begin{equation*}
\mathfrak{g} \equiv A_{1, p}(z) \mathfrak{g}^{p^{2}}+A_{2, p}(z) \mathfrak{g}^{p^{4}} \bmod p \tag{2.2}
\end{equation*}
$$

and the heights of $A_{1, p}(z) \bmod p$ and $A_{2, p}(z) \bmod p$ are less than $2 p^{4}$.
Remark 2.5. - An explicit formula for each rational function appearing in Equation (2.1) can be obtained from Theorem 9.2. Further, Theorem 9.4 gives an explicit formula for each rational function appearing in Equation (2.2).

## 3. Proof of Theorem 1.3

The proof of Theorem 1.3 is based on Theorem 3.2 and Proposition 3.3, which are stated below. In order to formulate Theorem 3.2, we have to define the $\mathbf{P}_{p, l}$ property. We denote by $\mathbb{Z}_{(p)}^{*}$ the set of units of $\mathbb{Z}_{(p)}$. As we have already said, the ring $\mathbb{Z}_{(p)}$ is a local ring and its maximal ideal is $(p) \mathbb{Z}_{(p)}$. So, $\gamma \in \mathbb{Z}_{(p)}^{*}$ if and only if $\gamma \notin(p) \mathbb{Z}_{(p)}$ if and only if $v_{p}(\gamma)=0$.

Definition 3.1. - Let p be a prime number and let $\boldsymbol{\alpha}=\left(\alpha_{1}, \ldots, \alpha_{n}\right), \boldsymbol{\beta}=\left(\beta_{1}, \ldots, \beta_{n-1}, 1\right)$ be in $\left(\mathbb{Z}_{(p)}\right)^{n}$ and let $l \geqslant 1$ be an integer. We say that $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ satisfies the $\boldsymbol{P}_{p, l}$ property, if, for every $k \in\{1, \ldots, l\}$, we have:
(P1) $\mathfrak{D}_{p}^{k}(\boldsymbol{\alpha})$ and $\mathfrak{D}_{p}^{k}(\boldsymbol{\beta})$ belong to $\left(\mathbb{Z}_{(p)}^{*} \cap(0,1]\right)^{n}$,
(P2) $\mathfrak{D}_{p}^{k}\left(\alpha_{i}\right)-\mathfrak{D}_{p}^{k}\left(\beta_{j}\right)$ belongs to $\mathbb{Z}_{(p)}^{*}$ for $1 \leqslant i, j \leqslant n$,
(P3) $\mathfrak{D}_{p}^{k}\left(\beta_{j}\right)-\mathfrak{D}_{p}^{k}\left(\beta_{s}\right)$ belongs to $\mathbb{Z}_{(p)}^{*}$ if and only if $\beta_{j} \neq \beta_{s}$,
(P4) $p-1 \notin I_{\boldsymbol{\beta}}^{(k+1)}$, where $I_{\boldsymbol{\beta}}^{(k+1)}=\left\{p \mathfrak{D}_{p}^{k+1}\left(\beta_{j}\right)-\mathfrak{D}_{p}^{k}\left(\beta_{j}\right): 1 \leqslant j \leqslant n\right.$ and $\left.\beta_{j} \neq 1\right\}$.
(P5) For every, $i, j \in\{1, \ldots, n\}, 1-\mathfrak{D}_{p}^{k}\left(\beta_{j}\right)+\mathfrak{D}_{p}^{k}\left(\alpha_{i}\right) \in \mathbb{Z}_{(p)}^{*}$ and $1-\mathfrak{D}_{p}^{k}\left(\beta_{j}\right)+\mathfrak{D}_{p}^{k}\left(\beta_{i}\right) \in$ $\mathbb{Z}_{(p)}^{*}$.

We are now ready to state Theorem 3.2.

Theorem 3.2.-Let $\boldsymbol{\alpha}=\left(\alpha_{1}, \ldots, \alpha_{n}\right), \boldsymbol{\beta}=\left(\beta_{1}, \ldots, \beta_{n-1}, 1\right)$ be in $(\mathbb{Q} \cap(0,1])^{n}$ and let $p$ be a prime number such that $f(z):={ }_{n} F_{n-1}(\boldsymbol{\alpha}, \boldsymbol{\beta} ; z)$ belongs to $\mathbb{Z}_{(p)}[[z]]$. Suppose that $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ satisfies the $\boldsymbol{P}_{p, l}$ property, where $l$ is the order of $p$ in $\left(\mathbb{Z} / d_{\boldsymbol{\alpha}, \boldsymbol{\beta}} \mathbb{Z}\right)^{*}$. If, for all $i, j \in\{1, \ldots, n\}$, $\alpha_{i}$ and $\beta_{j}$ belong to $\mathbb{Z}_{(p)}^{*}$ then the assertions (1), (2), and (3) of Theorem 1.3 hold.

The next proposition deals with some properties of the map $\mathfrak{D}_{p}$.
Proposition 3.3. - Let $\gamma=a / b$ and $\tau=c / d$ be in $\mathbb{Q} \cap(0,1]$ written in lowest form and let $p$ be a prime number such that $\gamma, \tau \in \mathbb{Z}_{(p)}$. Then:
(1) $\mathfrak{D}_{p}(\gamma)=y / b \in \mathbb{Q} \cap(0,1]$, where $y \in\{1, \ldots, b\}$ and $p y \equiv a \bmod b$. Moreover, $y$ and $b$ are co-prime,
(2) $\mathfrak{D}_{p}(\gamma)=\mathfrak{D}_{p}(\tau)$ if and only if $\gamma=\tau$,
(3) if $p>b, \mathfrak{D}_{p}(\gamma) \in \mathbb{Z}_{(p)}^{*}$.

Proof. - (1). By definition of $\mathfrak{D}_{p}$, it follows that $p \mathfrak{D}_{p}(\gamma)=\gamma+s$ with $s \in\{0, \ldots, p-1\}$. So, $\gamma+s=(a+s b) / b \in(p) \mathbb{Z}_{(p)}$. Thus, $y:=(a+s b) / p \in \mathbb{N}$. So, $\mathfrak{D}_{p}(\gamma)=y / b$ and $p y \equiv a \bmod b$. Assume for contradiction that $y=0$. Then, $\mathfrak{D}_{p}(\gamma)=0$ and $a \in(b) \mathbb{Z}$. But, by hypotheses, $0<a / b \leqslant 1$. Thus, $a=b$ and so, $\gamma=1$. But, $\mathfrak{D}_{p}(1)=1$. Whence, $0=1$, which is a contradiction. Thus, $y>0$. We now prove that $y \in\{1, \ldots, b-1, b\}$. Since $\gamma \in(0,1]$ and $s \leqslant p-1$, it follows that $\gamma+s \leqslant p$. As $\gamma+s=(a+b s) / b$ then $(a+b s) / b \leqslant p$. Thus, $y=(a+b s) / p \leqslant b$. Consequently, $y \in\{1, \ldots, b-1, b\}$. Therefore, $\mathfrak{D}_{p}(\gamma) \in \mathbb{Q} \cap(0,1]$. Finally, we show that $y$ and $b$ are co-prime. As $p y \equiv a \bmod b$ and, by assumption, $a$ and $b$ are co-prime then $y$ and $b$ are co-prime.
(2). It is clear that if $\gamma=\tau$ then $\mathfrak{D}_{p}(\gamma)=\mathfrak{D}_{p}(\tau)$. We now prove that if $\mathfrak{D}_{p}(\gamma)=\mathfrak{D}_{p}(\tau)$ then $\gamma=\tau$. From (1), we know that $\mathfrak{D}_{p}(\gamma)=y / b \in \mathbb{Q} \cap(0,1]$, where $y \in\{1, \ldots, b\}$ and $p y \equiv a \bmod b$ and $\mathfrak{D}_{p}(\tau)=x / d \in \mathbb{Q} \cap(0,1]$, where $x \in\{1, \ldots, d\}$ and $p x \equiv c \bmod d$. First, we suppose that $\gamma=1$. So, $1=\mathfrak{D}_{p}(\gamma)=y / b$. In particular, $y=b$. By assumption, $\mathfrak{D}_{p}(\gamma)=\mathfrak{D}_{p}(\tau)$. Then, $1=\mathfrak{D}_{p}(\tau)$ and $x=d$. Thus, $p d \equiv c \bmod d$. Whence, $c \in(d) \mathbb{Z}$. But, by hypotheses, $0<c / d \leqslant 1$. For this reason, $c=d$. Therefore, $\tau=1$. Now, we suppose that $\gamma<1$. Assume for contradiction that $\mathfrak{D}_{p}(\gamma)=1$. Then, $y=b$ and $p b \equiv a \bmod b$. Whence, $a \in(b) \mathbb{Z}$. Since $0<a / b \leqslant 1$, we have $a=b$. Therefore, $\gamma=1$, which is a contradiction. Consequently, $\mathfrak{D}_{p}(\gamma)<1$. By assumption, $\mathfrak{D}_{p}(\gamma)=\mathfrak{D}_{p}(\tau)$. Then, $\mathfrak{D}_{p}(\tau)<1$. From (1), we know that $\mathfrak{D}_{p}(\gamma)=y / b$ where $y \in\{1, \ldots, b\}$ and $\mathfrak{D}_{p}(\tau)=x / d$, where $x \in\{1, \ldots, d\}$. Actually, we have $y \in\{1, \ldots, b-1\}$ and $x \in\{1, \ldots, d-1\}$ because $0<y / b, x / d<1$. But, $y / b=x / d$ because $\mathfrak{D}_{p}(\gamma)=\mathfrak{D}_{p}(\tau)$. As $y, b$ are co-prime and $x, d$ are co-prime then the equality $y / b=x / d$ implies $y=x$ and $b=d$. In particular, we have $p y \equiv a \bmod b$ and $p y \equiv c \bmod b$. So, $a-c \in(b) \mathbb{Z}$. As $\gamma=a / b<1, \tau=c / d \leqslant 1$, and $d=b$ then $|a-c|<b$. But, $a-c \in(b) \mathbb{Z}$. Thus, $a=c$. So, $\gamma=\tau$.
(3). According to (1), $\mathfrak{D}_{p}(\gamma)=y / b$, where $y \in\{1, \ldots, b\}$ and $y$ and $b$ are co-prime. Since $p>b$, we have $p>y$. In particular, $p$ does not divide $y$ and thus, $y / b \in \mathbb{Z}_{(p)}^{*}$.

By assuming Theorem 3.2, we can now prove Theorem 1.3.

Proof of Theorem 1.3. - Let $p>2 d_{\boldsymbol{\alpha}, \boldsymbol{\beta}}$ be a prime number. Then $\alpha_{i}, \beta_{j} \in \mathbb{Z}_{(p)}$ for all $1 \leqslant i, j \leqslant n$ given that $p>d_{\boldsymbol{\alpha}, \boldsymbol{\beta}}$ and $\alpha_{i}, \beta_{j} \in(0,1]$ for all $1 \leqslant i, j \leqslant n$. We first prove that the following two conditions are satisfied.
a) For every integer $m \geqslant 0, \mathfrak{D}_{p}^{m}\left(\alpha_{i}\right)-\mathfrak{D}_{p}^{m}\left(\beta_{j}\right) \notin \mathbb{Z}$ for all $1 \leqslant i, j \leqslant n$.
b) For every integer $m \geqslant 0, \mathfrak{D}_{p}^{m}\left(\beta_{j}\right)-\mathfrak{D}_{p}^{m}\left(\beta_{s}\right) \notin \mathbb{Z}$ if and only if $\beta_{j} \neq \beta_{s}$.

By hypotheses, we know that, for all $i, j \in\{1, \ldots, n\}, \alpha_{i}-\beta_{j} \notin \mathbb{Z}$. That is equivalent to saying that, $\alpha_{i} \neq \beta_{j}$ because, for all $i, j \in\{1, \ldots, n\}, \alpha_{i}, \beta_{j}$ belong to ( 0,1$]$. Thus, it follows from (2) of Proposition 3.3 that, for all integers $m \geqslant 0, \mathfrak{D}_{p}^{m}\left(\alpha_{i}\right) \neq \mathfrak{D}_{p}^{m}\left(\beta_{j}\right)$. Thus, $\mathfrak{D}_{p}^{m}\left(\alpha_{i}\right)-\mathfrak{D}_{p}^{m}\left(\beta_{j}\right) \notin \mathbb{Z}$ because, for all integers $m \geqslant 0, \mathfrak{D}_{p}^{m}\left(\alpha_{i}\right), \mathfrak{D}_{p}^{m}\left(\beta_{j}\right)$ belong to ( 0,1$]$. Therefore, the condition (a) is satisfied. Following the same argument, one shows that the condition (b) is also satisfied.

We now prove that $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ satisfies the $\mathbf{P}_{p, m}$ property for all integers $m \geqslant 1$. To this end, we set

$$
\text { - } \mathcal{U}_{1}=\left\{\mathfrak{D}_{p}^{m}\left(\alpha_{i}\right), \mathfrak{D}_{p}^{m}\left(\beta_{j}\right)\right\}_{m \geqslant 1,1 \leqslant i, j \leqslant n}
$$

As $p>d_{\boldsymbol{\alpha}, \boldsymbol{\beta}}$ and $\alpha_{i}, \beta_{j} \in(0,1]$ for all $1 \leqslant i, j \leqslant n$ then $\alpha_{i}, \beta_{j} \in \mathbb{Z}_{(p)}^{*}$ for all $1 \leqslant i, j \leqslant n$. Then, it follows from (1) and (3) of Proposition 3.3 that $\mathcal{U}_{1} \subset \mathbb{Z}_{(p)}^{*} \cap(0,1]$.

Now, we consider the following set,

- $\mathcal{U}_{2}=\left\{\mathfrak{D}_{p}^{m}\left(\alpha_{i}\right)-\mathfrak{D}_{p}^{m}\left(\beta_{j}\right)\right\}_{m \geqslant 1,1 \leqslant i, j \leqslant n}$.

We have $0 \notin \mathcal{U}_{2}$ because, by condition a), we know that, for every $m \geqslant 1, \mathfrak{D}_{p}^{m}\left(\alpha_{i}\right)-\mathfrak{D}_{p}^{m}\left(\beta_{j}\right) \notin$ $\mathbb{Z}$ for all $1 \leqslant i, j \leqslant n$. Now, we prove that for any $1 \leqslant i, j \leqslant n$, $\mathfrak{D}_{p}^{m}\left(\alpha_{i}\right)-\mathfrak{D}_{p}^{m}\left(\beta_{j}\right)$ belongs to $\mathbb{Z}_{(p)}^{*}$. Indeed, let $i, j$ be in $\{1, \ldots, n\}$ and let us write $\alpha_{i}=a / b$ and $\beta_{j}=c / d$ in lowest form. Then, from (1) of Proposition 3.3, we get $\mathfrak{D}_{p}^{m}\left(\alpha_{i}\right)=y / b$ and $\mathfrak{D}_{p}^{m}\left(\beta_{j}\right)=y^{\prime} / d$, where $0<y \leqslant b, 0<y^{\prime} \leqslant d$ and $y, b$ are co-prime and $y^{\prime}, d$ are co-prime. So, $\mathfrak{D}_{p}^{m}\left(\alpha_{i}\right)-\mathfrak{D}_{p}^{m}\left(\beta_{j}\right)=$ $\left(y d-y^{\prime} b\right) / b d$ and $\left|y d-y^{\prime} b\right|<b d$. As $p>d_{\boldsymbol{\alpha}, \boldsymbol{\beta}}$ then $p>b d$ and thus, $p>\left|y d-y^{\prime} b\right|$. So, $\mathfrak{D}_{p}^{m}\left(\alpha_{i}\right)-\mathfrak{D}_{p}^{m}\left(\beta_{j}\right)$ belongs to $\mathbb{Z}_{(p)}^{*}$. Thus, $\mathcal{U}_{2} \subset \mathbb{Z}_{(p)}^{*}$.

We also consider the following set,

- $\mathcal{U}_{3}=\left\{\mathfrak{D}_{p}^{m}\left(\beta_{j}\right)-\mathfrak{D}_{p}^{m}\left(\beta_{s}\right): 1 \leqslant j, s \leqslant n, \beta_{j} \neq \beta_{s}\right\}_{m \geqslant 1}$.

We have $0 \notin \mathcal{U}_{3}$ because, by condition b), we know that, for every $m \geqslant 1, \mathfrak{D}_{p}^{m}\left(\beta_{j}\right)-\mathfrak{D}_{p}^{m}\left(\beta_{s}\right) \notin$ $\mathbb{Z}$ if and only if $\beta_{j} \neq \beta_{s}$. Following the same argument as in $\mathcal{U}_{2}$, one gets $\mathcal{U}_{3} \subset \mathbb{Z}_{(p)}^{*}$.

We have the following set,

- $\mathcal{U}_{4}=\left\{1-\mathfrak{D}_{p}^{m}\left(\beta_{j}\right): 1 \leqslant j \leqslant n, \beta_{j} \neq 1\right\}_{m \geqslant 1}$.

Assume for contradiction that $0 \in \mathcal{U}_{4}$. Then $1=\mathfrak{D}_{p}^{m}\left(\beta_{j}\right)$ for somme $j \in\{1, \ldots, n\}$ with $\beta_{j} \neq 1$. As $\mathfrak{D}_{p}^{m}(1)=1=\mathfrak{D}_{p}^{m}\left(\beta_{j}\right)$ then, according to (2) of Proposition 3.3, $\beta_{j}=1$, which is a contradiction. Therefore, $0 \notin \mathcal{U}_{4}$. Following the same argument as in $\mathcal{U}_{2}$, one gets $\mathcal{U}_{4} \subset \mathbb{Z}_{(p)}^{*}$.

- $\mathcal{U}_{5}=\left\{1-\mathfrak{D}_{p}^{m}\left(\beta_{j}\right)+\mathfrak{D}_{p}^{m}\left(\alpha_{i}\right), 1-\mathfrak{D}_{p}^{m}\left(\beta_{t}\right)+\mathfrak{D}_{p}^{m}\left(\beta_{s}\right)\right\}_{m \geqslant 1,1 \leqslant i, j, t, s \leqslant n}$.

We have $0 \notin \mathcal{U}_{5}$ because, from (1) of Proposition 3.3, for every $m \geqslant 1, \mathfrak{D}_{p}^{m}(\boldsymbol{\alpha})$ and $\mathfrak{D}_{p}^{m}(\boldsymbol{\beta})$ belong to $((0,1] \cap \mathbb{Q})^{n}$. We now prove that $\mathcal{U}_{5} \subset \mathbb{Z}_{(p)}^{*}$. Indeed, let $i, j$ be in $\{1, \ldots, n\}$ and let us write $\alpha_{i}=a / b$ and $\beta_{j}=c / d$ in lowest form. Then, from (1) of Proposition 3.3, we get $\mathfrak{D}_{p}^{m}\left(\alpha_{i}\right)=y / a$ and $\mathfrak{D}_{p}^{m}\left(\beta_{j}\right)=y^{\prime} / d$, where $0<y \leqslant b, 0<y^{\prime} \leqslant d$ and $y, b$ are co-prime and $y^{\prime}, d$ are co-prime. So, $1-\mathfrak{D}_{p}^{m}\left(\beta_{j}\right)+\mathfrak{D}_{p}^{m}\left(\alpha_{i}\right)=\left(b d-y^{\prime} b+y d\right) / b d$ and $\left|b d-y^{\prime} b+y d\right|<2 b d$. As $p>2 d_{\boldsymbol{\alpha}, \boldsymbol{\beta}}$ then $p>2 b d$ and thus, $p>\left|b d-y^{\prime} b+y d\right|$. So, $1-\mathfrak{D}_{p}^{m}\left(\beta_{j}\right)+\mathfrak{D}_{p}^{m}\left(\alpha_{i}\right) \in \mathbb{Z}_{(p)}^{*}$. In a similar way, one shows that, for any $1 \leqslant m, s \leqslant n, 1-\mathfrak{D}_{p}^{m}\left(\beta_{t}\right)+\mathfrak{D}_{p}^{m}\left(\beta_{s}\right) \in \mathbb{Z}_{(p)}^{*}$.

We now see that $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ satisfies the $\mathbf{P}_{p, m}$ property. The condition $(\mathbf{P} 1)$ is satisfied because $\mathcal{U}_{1} \subset \mathbb{Z}_{(p)}^{*} \cap(0,1]$. Now, since $\mathcal{U}_{2} \subset \mathbb{Z}_{(p)}^{*}$, the condition $(\mathbf{P} 2)$ is satisfied. The condition $(\mathbf{P} 3)$ is also satisfied because $\mathcal{U}_{3} \subset \mathbb{Z}_{(p)}^{*}$. Assume now for contradiction that $p-1 \in I_{\boldsymbol{\beta}}^{(k+1)}$ for some $k \in\{1, \ldots, l\}$. Then, $p-1=p \mathfrak{D}_{p}^{k+1}\left(\beta_{j}\right)-\mathfrak{D}_{p}^{k}\left(\beta_{j}\right)$ for $\beta_{j} \neq 1$. So that, $1-\mathfrak{D}_{p}^{k}\left(\beta_{j}\right) \in p \mathbb{Z}_{(p)}$. But, $1-\mathfrak{D}_{p}^{k}\left(\beta_{j}\right) \in \mathbb{Z}_{(p)}^{*}$ because $1-\mathfrak{D}_{p}^{k}\left(\beta_{j}\right) \in \mathcal{U}_{4}$. So, we obtain a contradiction. Thus, for all $k \in\{1, \ldots, l\}, p-1 \notin I_{\boldsymbol{\beta}}^{(k+1)}$. Whence, the condition (P4) is satisfied. Finally, the condition ( $\mathbf{P} 5$ ) is satisfied since $\mathcal{U}_{5} \subset \mathbb{Z}_{(p)}^{*}$. Hence, $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ satisfies the $\mathbf{P}_{p, m}$ property for all integers $m \geqslant 1$. In particular, $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ satisfies the $\mathbf{P}_{p, l}$ property, where $l$ is the order of $p$ in $\left(\mathbb{Z} / d_{\boldsymbol{\alpha}, \boldsymbol{\beta}} \mathbb{Z}\right)^{*}$.

We have already seen that, for all $i, j \in\{1, \ldots, n\}, \alpha_{i}$ and $\beta_{j}$ belong to $\mathbb{Z}_{(p)}^{*}$. Consequently, by applying Theorem 3.2, the assertions (1), (2), and (3) hold, which completes the proof.

Remark 3.4. - Let $\boldsymbol{\alpha}=\left(\alpha_{1}, \ldots, \alpha_{n}\right), \boldsymbol{\beta}=\left(\beta_{1}, \ldots, \beta_{n-1}, 1\right)$ be in $(\mathbb{Q} \cap(0,1])^{n}, p>$ $2 d_{\boldsymbol{\alpha}, \boldsymbol{\beta}}$ be a prime number. Then, it follows from the proof of Theorem 1.3 that $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ satisfies the $\mathbf{P}_{p, m}$ property for all integers $m \geqslant 1$.

## 4. Proof of Theorem 3.2

Theorem 3.2 is derived from Propositions 4.1 and 4.2. In order to state Proposition 4.1, we need to introduce two more sets and some notations. Let $p$ be a prime number. For $\gamma=\left(\gamma_{1}, \ldots, \gamma_{n}\right) \in \mathbb{Z}_{(p)}^{n}$ and $r \in \mathbb{Z}_{\geqslant 0}$, we consider the following two sets:

$$
\mathcal{P}_{\gamma, r}=\left\{s \in\{1, \ldots, n\}:\left(\gamma_{s}\right)_{r} \in p \mathbb{Z}_{(p)}\right\} \text { and } \mathcal{C}_{\gamma, r}=\left\{s \in\{1, \ldots, n\}:\left(\gamma_{s}\right)_{r} \notin p \mathbb{Z}_{(p)}\right\} .
$$

Note that $\mathcal{C}_{\boldsymbol{\gamma}, r}$ is the complement of $\mathcal{P}_{\boldsymbol{\gamma}, r}$ in $\{1, \ldots, n\}$ and that $\mathcal{C}_{\gamma, 0}=\{1, \ldots, n\}$.
Let $\boldsymbol{\alpha}=\left(\alpha_{1}, \ldots, \alpha_{n}\right), \boldsymbol{\beta}=\left(\beta_{1}, \ldots, \beta_{n-1}, 1\right)$ be in $\left(\mathbb{Q} \backslash \mathbb{Z}_{\leqslant 0}\right)^{n}$ and let $p$ be a prime number such that $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ belong to $\mathbb{Z}_{(p)}^{n}$. Then, for every integer $a \geqslant 1$ and for every $r \in\{0, \ldots, p-1\}$, we set $\boldsymbol{\alpha}_{a, r}=\left(\alpha_{1, a, r}, \ldots, \ldots, \alpha_{n, a, r}\right)$ and $\boldsymbol{\beta}_{a, r}=\left(\beta_{1, a, r}, \ldots, \ldots, \beta_{n, a, r}\right)$, where, for every $s \in\{1, \ldots, n\}$,

$$
\alpha_{s, a, r}=\left\{\begin{array}{lll}
\mathfrak{D}_{p}^{a}\left(\alpha_{s}\right) & \text { if } & s \in \mathcal{C}_{\mathfrak{D}_{p}^{a-1}(\boldsymbol{\alpha}), r} \\
\mathfrak{D}_{p}^{a}\left(\alpha_{s}\right)+1 & \text { if } & s \in \mathcal{P}_{\mathfrak{D}_{p}^{a-1}(\boldsymbol{\alpha}), r},
\end{array}\right.
$$

$$
\beta_{s, a, r}=\left\{\begin{array}{lll}
\mathfrak{D}_{p}^{a}\left(\beta_{s}\right) & \text { if } & s \in \mathcal{C}_{\mathfrak{D}_{p}^{a-1}(\boldsymbol{\beta}), r} \\
\mathfrak{D}_{p}^{a}\left(\beta_{s}\right)+1 & \text { if } & s \in \mathcal{P}_{\mathfrak{D}_{p}^{a-1}(\boldsymbol{\beta}), r}
\end{array}\right.
$$

Note that, for every $a \geqslant 1$ and $r \in\{0, \ldots, p-1\}, \beta_{n, a, r}=1$ because $n \in \mathcal{C}_{\mathfrak{D}_{p}^{a-1}(\boldsymbol{\beta}), r}$ and $\mathfrak{D}_{p}^{a}(1)=1$. So, it makes sense to consider the hypergeometric series ${ }_{n} F_{n-1}\left(\boldsymbol{\alpha}_{a, r}, \boldsymbol{\beta}_{a, r} ; z\right)$. We let $f_{a, r}$ denote the hypergeometric series ${ }_{n} F_{n-1}\left(\boldsymbol{\alpha}_{a, r}, \boldsymbol{\beta}_{a, r} ; z\right)$. Thus,

$$
f_{a, r}=\sum_{m \geqslant 0}\left(\frac{\prod_{s \in \mathcal{C}_{\mathfrak{O}_{p}^{a-1}(\boldsymbol{\alpha}), r}}\left(\mathfrak{D}_{p}^{a}\left(\alpha_{s}\right)\right)_{m} \prod_{s \in \mathcal{P}_{\mathfrak{O}_{p}^{a-1}(\boldsymbol{\alpha}), r}}\left(\mathfrak{D}_{p}^{a}\left(\alpha_{s}\right)+1\right)_{m}}{\prod_{s \in \mathcal{C}_{\mathfrak{O}_{p}^{a-1}(\boldsymbol{\beta}), r}}\left(\mathfrak{D}_{p}^{a}\left(\beta_{s}\right)\right)_{m} \prod_{s \in \mathcal{P}_{\mathfrak{O}_{p}^{a-1}(\boldsymbol{\beta}), r}}\left(\mathfrak{D}_{p}^{a}\left(\beta_{s}\right)+1\right)_{m}}\right) z^{m}
$$

Proposition 4.1. - Let the assumptions be as in Theorem 3.2. Then, for every $r \in$ $S_{\mathfrak{D}_{p}^{l-1}(\boldsymbol{\alpha}), \mathfrak{D}_{p}^{l-1}(\boldsymbol{\beta}), p}, f_{l, r} \in 1+z \mathbb{Z}_{(p)}[[z]]$ and

$$
f_{l, r} \equiv \sum_{j \in S_{\mathfrak{O}_{p}^{l-1}(\boldsymbol{\alpha}), \mathcal{D}_{p}^{l-1}(\boldsymbol{\beta}), p}} Q_{r, j}(z) f_{l, j}^{p^{l}} \bmod p
$$

where, for every $j \in S_{\mathfrak{D}_{p}^{l-1}(\boldsymbol{\alpha}), \mathfrak{D}_{p}^{l-1}(\boldsymbol{\beta}), p}, Q_{r, j}(z)$ belongs to $\mathbb{Z}_{(p)}[z]$ and has degree less than $p^{l}$.

Proposition 4.2. - Let $g_{0}, g_{1}, \ldots, g_{t-r}$ be in $\mathbb{F}_{p}[[z]]$ different from zero and let $l$ be $a$ positive integer. Suppose that, for every $i \in\{0, \ldots, t-r\}$,

$$
g_{i}=\sum_{k=1}^{s} P_{i, k} g_{0}^{p^{k l}}+\sum_{k=1}^{t-r} A_{i, k} g_{k}^{p^{s l}}
$$

where, for all $i \in\{0, \ldots, t-r\}, P_{i, 1}, \ldots, P_{i, s}, A_{i, 1}, \ldots A_{i, t-r}$ belong to $\mathbb{F}_{p}(z)$ and their heights are less than $c p^{s l}$. If $A_{0, t-r}$ is not zero then, for every $i \in\{0, \ldots, t-r-1\}$,

$$
g_{i}=\sum_{k=1}^{2 s} T_{i, k} g_{0}^{p^{k l}}+\sum_{k=1}^{t-r-1} D_{i, k} g_{k}^{p^{2 s l}}
$$

where, for all $i \in\{0, \ldots, t-r-1\}, T_{i, 1}, \ldots, T_{i, 2 s}, D_{i, 1}, \ldots D_{i, t-r-1}$ belong to $\mathbb{F}_{p}(z)$ and their heights are less than $5 c(t-r+1) p^{2 s l}$.

By assuming Propositions 4.1 and 4.2 , we are now in a position to prove Theorem 3.2.
Proof of Theorem 3.2. - Note that $f_{l, 0}$ is the hypergeometric series ${ }_{n} F_{n-1}(\boldsymbol{\alpha}, \boldsymbol{\beta} ; z)$ because, by assumption $\boldsymbol{\alpha}, \boldsymbol{\beta}$ belong to $\left(\mathbb{Z}_{(p)}^{*} \cap(0,1]\right)^{n}$ and thus, Lemma 2.1 implies $\mathfrak{D}_{p}^{l}(\boldsymbol{\alpha})=$ $\boldsymbol{\alpha}$ and $\mathfrak{D}_{p}^{l}(\boldsymbol{\beta})=\boldsymbol{\beta}$.
1). If $\# S_{\mathfrak{D}_{p}^{l-1}(\boldsymbol{\alpha}), \mathfrak{D}_{p}^{l-1}(\boldsymbol{\beta}), p}=1$ then, by Proposition 4.1, we have

$$
f_{l, 0}(z) \equiv Q_{0,0} f_{l, 0}^{p^{l}} \bmod p
$$

where $Q_{0,0}$ is a polynomial with coefficients in $\mathbb{Z}_{(p)}$ whose degree is less than $p^{l}$.
2). Suppose that $\# S_{\mathfrak{D}_{p}^{l-1}(\boldsymbol{\alpha}), \mathfrak{D}_{p}^{l-1}(\boldsymbol{\beta}), p}=2$. Let us write $S_{\mathfrak{D}_{p}^{l-1}(\boldsymbol{\alpha}), \mathfrak{D}_{p}^{l-1}(\boldsymbol{\beta}), p}=\left\{r_{0}, r_{1}\right\}$ with $r_{0}=0$. Then, by Proposition 4.1, we have

$$
\begin{align*}
f_{l, 0} & \equiv Q_{0,0} f_{l, 0}^{p^{l}}+Q_{0,1} f_{l, r_{1}}^{p^{l}} \bmod p  \tag{4.1}\\
f_{l, r_{1}} & \equiv Q_{1,0} f_{l, 0}^{p^{l}}+Q_{1,1} f_{l, r_{1}}^{p^{l}} \bmod p \tag{4.2}
\end{align*}
$$

where $Q_{0,0}(z), Q_{0,1}(z), Q_{1,0}(z)$ and $Q_{1,1}(z)$ belong to $\mathbb{Z}_{(p)}[z]$ and their degrees are less than $p^{l}$.

If $Q_{0,1}(z) \bmod p$ is the zero polynomial then, from (4.1), we have $f_{l, 0} \equiv Q_{0,0} f_{l, 0}^{p^{l}} \bmod p$. Now, suppose that $Q_{0,1}(z) \bmod p$ is not the zero polynomial. From Equations (4.1) and (4.2), we have

$$
Q_{0,1} f_{l, r_{1}}-Q_{1,1} f_{l, 0} \equiv\left(Q_{0,1} Q_{1,0}-Q_{1,1} Q_{0,0}\right) f_{l, 0}^{p^{l}} \bmod p
$$

Since $Q_{0,1}(z) \bmod p$ is not the zero polynomial, it follows from the previous equality that

$$
f_{l, r_{1}} \equiv \frac{Q_{0,1} Q_{1,0}-Q_{1,1} Q_{0,0}}{Q_{0,1}} f_{l, 0}^{p^{l}}+\frac{Q_{1,1}}{Q_{0,1}} f_{l, 0} \bmod p
$$

As the characteristic of $\mathbb{F}_{p}$ is $p$, then

$$
f_{l, r_{1}}^{p^{l}} \equiv\left(\frac{Q_{0,1} Q_{1,0}-Q_{1,1} Q_{0,0}}{Q_{0,1}}\right)^{p^{l}} f_{l, 0}^{p^{2 l}}+\left(\frac{Q_{1,1}}{Q_{0,1}}\right)^{p^{l}} f_{l, 0}^{p^{l}} \bmod p
$$

By replacing the previous equality into (4.1), we obtain

$$
\begin{aligned}
f_{l, 0} & \equiv Q_{0,0} f_{l, 0}^{p^{l}}+Q_{0,1}\left(\left(\frac{Q_{0,1} Q_{1,0}-Q_{1,1} Q_{0,0}}{Q_{0,1}}\right)^{p^{l}} f_{l, 0}^{p^{2 l}}+\left(\frac{Q_{1,1}}{Q_{0,1}}\right)^{p^{l}} f_{l, 0}^{p^{l}}\right) \bmod p \\
& \equiv\left(Q_{0,0}+Q_{0,1}\left(\frac{Q_{1,1}}{Q_{0,1}}\right)^{p^{l}}\right) f_{l, 0}^{p^{l}}+Q_{0,1}\left(\frac{Q_{0,1} Q_{1,0}-Q_{1,1} Q_{0,0}}{Q_{0,1}}\right)^{p^{l}} f_{l, 0}^{p^{2 l}} \bmod p .
\end{aligned}
$$

Since the degrees of $Q_{0,0}, Q_{0,1}, Q_{1,0}$ and $Q_{1,1}$ are less than or equal to $p^{l}-1$, we conclude that the height of $Q_{0,0}+Q_{0,1}\left(\frac{Q_{1,1}}{Q_{0,1}}\right)^{p^{l}}$ is less than $p^{2 l}$ and that the height of the rational function $Q_{0,1}\left(\frac{Q_{0,1} Q_{1,0}-Q_{1,1} Q_{0,0}}{Q_{0,1}}\right)^{p^{l}}$ is less than $2 p^{2 l}$.
3). Suppose thta $t+1=\# S_{\mathfrak{D}_{p}^{l-1}(\boldsymbol{\alpha}), \mathfrak{D}_{p}^{l-1}(\boldsymbol{\beta}), p}$ with $t>1$. Let us write $S_{\mathfrak{D}_{p}^{l-1}(\boldsymbol{\alpha}), \mathfrak{D}_{p}^{l-1}(\boldsymbol{\beta}), p}=$ $\left\{r_{0}, r_{1}, \ldots, r_{t}\right\}$ with $r_{0}=0$. Then, by Proposition 4.1, we have for all $i \in\{0, \ldots, t\}$,

$$
\begin{equation*}
f_{l, r_{i}} \equiv Q_{i, 0} f_{l, 0}^{p^{l}}+\sum_{j=1}^{t} Q_{i, j}(z) f_{l, r_{j}}^{p^{l}} \bmod p \tag{4.3}
\end{equation*}
$$

where, for every $j \in\{0, \ldots, t\}, Q_{i, j}(z)$ is a polynomial with coefficients in $\mathbb{Z}_{(p)}$ whose degree is less than $p^{l}$.

If, for every $j \in\{1, \ldots, t\}, Q_{0, j}(z) \bmod p$ is the zero polynomial then, it follows from (4.3) that $f_{l, 0} \equiv Q_{0,0}(z) f_{l, 0}^{p^{l}} \bmod p$. Now, suppose that there is $j \in\{1, \ldots, t\}$ such that $Q_{0, j} \bmod p$ is not the zero polynomial. Without losing any generality we can assume that
$Q_{0, t} \bmod p$ is not the zero polynomial. Then, by applying Proposition 4.2 to (4.3), it follows that, for all $i \in\{0, \ldots, t-1\}$,

$$
\begin{equation*}
f_{l, r_{i}} \equiv \sum_{k=1}^{2} T_{i, k, 2} f_{l, 0}^{p^{k l}}+\sum_{k=1}^{t-1} D_{i, k, 2} f_{l, r_{k}}^{p^{2 l}} \bmod p \tag{4.4}
\end{equation*}
$$

where, $T_{i, 1,2}, T_{i, 2,2}, D_{i, 1,2}, \ldots D_{i, t-1,2}$ belong to $\mathbb{Q}(z) \cap \mathbb{Z}_{(p)}[[z]]\left[z^{-1}\right]$ and their heights are less than $5(t+1) p^{2 l}$.

Now, if for all $k \in\{1, \ldots, t-1\}, D_{0, k, 2} \bmod p=0$ then, $f_{l, 0} \equiv P_{1} f_{l, 0}^{p^{l}}+P_{2} f_{l, 0}^{p^{2 l}} \bmod p$, where $P_{1}=T_{0,1,2}$ and $P_{2}=T_{0,2,2}$.

Now, suppose that there is $k \in\{1, \ldots, t-1\}$ such that $D_{0, k, 2} \bmod p$ is not the zero polynomial. Without losing any generality we can assume that $D_{0, t-1,2} \bmod p$ is not the zero polynomial. Then, by applying Proposition 4.2 to (4.4), we infer that, for all $i \in\{0, \ldots, t-2\}$,

$$
\begin{equation*}
f_{l, r_{i}} \equiv \sum_{k=1}^{4} T_{i, k, 3} f_{l, 0}^{p^{k l}}+\sum_{k=1}^{t-2} D_{i, k, 3} f_{l, r_{k}}^{p^{4 l}} \bmod p \tag{4.5}
\end{equation*}
$$

where, $T_{i, 1,3}, \ldots, T_{i, 4,3}, D_{i, 1,3}, \ldots D_{i, t-2,3}$ belong to $\mathbb{Q}(z) \cap \mathbb{Z}_{(p)}[[z]]\left[z^{-1}\right]$ and their heights are less than $5^{2}(t+1) t p^{4 l}$.

After making the previous process $t$-times we deduce that,

$$
f_{l, 0} \equiv Q_{1} f_{l, 0}^{p^{l}}+Q_{2} f_{l, 0}^{p^{2 l}}+\cdots+Q_{2^{t}} f_{l, 0}^{p^{2^{t_{l}}}} \bmod p
$$

where, for every $i \in\{1, \ldots, t\}, Q_{i}$ belongs to $\mathbb{Q}(z) \cap \mathbb{Z}_{(p)}[[z]]\left[z^{-1}\right]$ and the height of $Q_{i} \bmod p$ is less than $5^{t}(t+1)!p^{2^{t} l}$.

## 5. Proof of Proposition 4.2

Proof. - By hypotheses, for every $i \in\{0, \ldots, t-r\}$, we have

$$
\begin{equation*}
g_{i}=\sum_{k=1}^{s} P_{i, k} g_{0}^{p^{k l}}+\sum_{k=1}^{t-r} A_{i, k} g_{k}^{p^{s l}} . \tag{5.1}
\end{equation*}
$$

Then, for every $i \in\{1, \ldots, t-r\}$,

$$
A_{0, t-r} g_{i}-A_{i, t-r} g_{0}=\sum_{k=1}^{s}\left(A_{0, t-r} P_{i, k}-A_{i, t-r} P_{0, k}\right) g_{0}^{p^{k l}}+\sum_{k=1}^{t-r-1}\left(A_{0, t-r} A_{i, k}-A_{i, t-r} A_{0, k}\right) g_{k}^{p^{s l}}
$$

By assumption, $A_{0, t-r}$ is not zero. Then, it follows from the last equality that, for every $i \in\{1, \ldots, t-r\}$,

$$
g_{i}=\sum_{k=1}^{s} Q_{i, k} g_{0}^{p^{k l}}+\sum_{k=1}^{t-r-1} B_{i, k} g_{k}^{p^{s l}}+C_{i} g_{0}
$$

where,

$$
Q_{i, k}=\frac{A_{0, t-r} P_{i, k}-A_{i, t-r} P_{0, k}}{A_{0, t-r}}, B_{i, k}=\frac{A_{0, t-r} A_{i, k}-A_{i, t-r} A_{0, k}}{A_{0, t-r}}, \text { and } C_{i}=\frac{A_{i, t-r}}{A_{0, t-r}} .
$$

As the characteristic of $\mathbb{F}_{p}$ is $p$ then, for every $i \in\{1, \ldots, t-r\}$,

$$
g_{i}^{p^{s l}}=\sum_{k=1}^{s} Q_{i, k}^{p^{s l}} g_{0}^{p^{(s+k) l}}+\sum_{k=1}^{t-r-1} B_{i, k}^{p^{s l}} g_{k}^{p^{2 s l}}+C_{i}^{p^{s l}} g_{0}^{p^{s l}} .
$$

By substituting this last equality into (5.1), for every $i \in\{0, \ldots, t-r-1\}$, we get

$$
\begin{aligned}
g_{i}= & \sum_{k=1}^{s-1} P_{i, k} g_{0}^{p^{k l}}+\left(P_{i, s}+\sum_{k=1}^{t-r} A_{i, k} C_{k}^{p^{s l}}\right) g_{0}^{p^{s l}}+\sum_{k=1}^{s}\left(\sum_{j=1}^{t-r} A_{i, j} Q_{j, k}^{p^{s l}}\right) g_{0}^{p^{(s+k) l}} \\
& +\sum_{k=1}^{t-r-1}\left(\sum_{j=1}^{t-r} A_{i, j} B_{j, k}^{p^{s l}}\right) g_{k}^{p^{2 s l}} .
\end{aligned}
$$

For every $k \in\{1, \ldots, s-1\}$, we set $T_{i, k}=P_{i, k}$, for $k=s$, we set $T_{i, s}=P_{i, s}+$ $\sum_{k=1}^{t-r} A_{i, k} C_{k}^{p^{s l}}$, for every $k \in\{s+1, \ldots, 2 s\}$, we set $T_{i, k}=\sum_{j=1}^{t-r} A_{i, j} Q_{j, k-s}^{p^{s l}}$, and finally for every $k \in\{1, \ldots, t-r-1\}$, we set $D_{i, k}=\sum_{j=1}^{t-r} A_{i, j} B_{j, k}^{p^{s l}}$. So that, for every $i \in\{0, \ldots, t-r-1\}$, we have

$$
g_{i}=\sum_{k=1}^{2 s} T_{i, k} g_{0}^{p^{k l}}+\sum_{k=1}^{t-r-1} D_{i, k} g_{k}^{p^{2 s l}} .
$$

Finally, we are going to see that, for all $i \in\{0, \ldots, t-r-1\}$, the heights of $T_{i, 1}, \ldots, T_{i, 2 s}$, $D_{i, 1}, \ldots D_{i, t-r-1}$ are less than $5 c(t-r+1) p^{2 s l}$. In fact, if $k \in\{1, \ldots, s-1\}$ then, $T_{i, k}=$ $P_{i, k}$. By hypotheses, the height of $P_{i, k}$ is less than $c p^{s l}$. So, if $k \in\{1, \ldots, s-1\}$ then the height of $T_{i, k}$ is less than $c p^{s l}$. By definition, $T_{i, s}=P_{i, s}+\sum_{k=1}^{t-r} A_{i, k} C_{k}^{p^{s l}}$. Recall that, for every $k \in\{1, \ldots, t-r-1\}, C_{k}=\frac{A_{k, t-r}}{A_{0, t-r}}$. Thus, the height of $C_{k}$ is less than $2 c p^{s l}$ because, by hypotheses, the heights of $A_{k, t-r}$ and $A_{0, t-r}$ are less than $c p^{s l}$. So, for every $k \in\{1, \ldots, t-r-1\}$, the height of $C_{k}^{p^{s l}}$ is less than $2 c p^{2 s l}$. Again, by hypotheses, the height of $A_{i, k}$ is $c p^{s l}$. Thus, the height of $A_{i, k} C_{k}^{p^{s l}}$ is less than $3 c p^{2 s l}$. Thus, the height of $T_{i, s}$ is less than $3 c(t-r+1) p^{2 s l}$. Now, we prove that, for every $k \in\{s+1, \ldots, 2 s\}$, the height of $T_{i, k}$ is less than $5 c(t-r) p^{2 s l}$. By definition, $T_{i, k}=\sum_{j=1}^{t-r} A_{i, j} Q_{j, k-s}^{p^{s l}}$. The height of $Q_{j, k-s}$ is less than $4 c p^{s l}$ because, $Q_{j, k-s}=P_{j, k-s}-\left(A_{j, t-r} P_{0, k-s}\right) /\left(A_{0, t-r}\right)$ and by hypotheses, the heights of $A_{0, t-r}, A_{j, t-r}, P_{j, k-s}, P_{0, k-s}$ are less that $c p^{s l}$. Thus, the height of $Q_{j, k-s}^{p^{s l}}$ is less than $4 c p^{2 s l}$. So, the height of $A_{i, j} Q_{j, k-s}^{p^{s l}}$ is less than $5 c p^{2 s l}$. Whence, the height of $T_{i, k}$ is less than $5 c(t-r) p^{2 s l}$. Similarly, it follows that, for every $k \in\{1, \ldots, t-r-1\}$, the height of $D_{i, k}$ is $5 c(t-r) p^{2 s l}$. This completes the proof of our proposition.

## 6. Proof of Proposition 4.1

The proof of Proposition 4.1 relies on Lemmas 6.1 and 6.2.
Lemma 6.1. - Let $\boldsymbol{\alpha}=\left(\alpha_{1}, \ldots, \alpha_{n}\right), \boldsymbol{\beta}=\left(\beta_{1}, \ldots, \beta_{n-1}, 1\right)$ be in $\left(\mathbb{Q} \backslash \mathbb{Z}_{\leqslant 0}\right)^{n}$ and let $p$ be a prime number such that $p$ does not divide $d_{\boldsymbol{\alpha}, \boldsymbol{\beta}}$. Suppose that $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ satisfies the $\boldsymbol{P}_{p, l}$ property. Then:
A) for every $(a, r) \in\{1, \ldots, l\} \times\{0, \ldots, p-1\}$, the map $\sigma: S_{\boldsymbol{\alpha}_{a, r}, \boldsymbol{\beta}_{a, r}, p} \rightarrow S_{\mathfrak{D}_{p}^{a}(\boldsymbol{\alpha}), \mathfrak{D}_{p}^{a}(\boldsymbol{\beta}), p}$ given by

$$
\sigma(t)= \begin{cases}t & \text { if } \quad t \equiv 1-\mathfrak{D}_{p}^{a}\left(\beta_{s}\right) \bmod p \text { with } s \in \mathcal{C}_{\mathfrak{D}_{p}^{a-1}(\boldsymbol{\beta}), r} \\ t+1 & \text { if } \quad t \equiv-\mathfrak{D}_{p}^{a}\left(\beta_{s}\right) \bmod p \text { with } s \in \mathcal{P}_{\mathfrak{D}_{p}^{a-1}(\boldsymbol{\beta}), r}\end{cases}
$$

is well-defined and is bijective. Moreover, its inverse $\tau: S_{\mathfrak{D}_{p}^{a}(\boldsymbol{\alpha}), \mathfrak{D}_{p}^{a}(\boldsymbol{\beta}), p} \rightarrow S_{\boldsymbol{\alpha}_{a, r}, \boldsymbol{\beta}_{a, r}, p}$ is given by

$$
\tau(t)=\left\{\begin{array}{lll}
t & \text { if } & t \equiv 1-\mathfrak{D}_{p}^{a}\left(\beta_{s}\right) \bmod p \text { with } s \in \mathcal{C}_{\mathfrak{D}_{p}^{a-1}(\boldsymbol{\beta}), r} \\
t-1 & \text { if } & t \equiv 1-\mathfrak{D}_{p}^{a}\left(\beta_{s}\right) \bmod p \text { with } s \in \mathcal{P}_{\mathfrak{D}_{p}^{a-1}(\boldsymbol{\beta}), r}
\end{array}\right.
$$

B) for every $(a, r) \in\{1, \ldots, l\} \times\{0, \ldots, p-1\}$, the following equalities hold for every $t \in S_{\boldsymbol{\alpha}_{a, r}, \boldsymbol{\beta}_{a, r}, p}, \mathcal{P}_{\boldsymbol{\alpha}_{a, r}, t}=\mathcal{P}_{\mathfrak{D}_{p}^{a}(\boldsymbol{\alpha}), \sigma(t)}, \mathcal{C}_{\boldsymbol{\alpha}_{a, r}, t}=\mathcal{C}_{\mathfrak{D}_{p}^{a}(\boldsymbol{\alpha}), \sigma(t)}, \mathcal{P}_{\boldsymbol{\beta}_{a, r}, t}=\mathcal{P}_{\mathfrak{D}_{p}^{a}(\boldsymbol{\beta}), \sigma(t)}$, and $\mathcal{C}_{\boldsymbol{\beta}_{a, r}, t}=\mathcal{C}_{\mathfrak{D}_{p}^{a}(\boldsymbol{\beta}), \sigma(t)}$.
Lemma 6.2. - Let $\boldsymbol{\alpha}=\left(\alpha_{1}, \ldots, \alpha_{n}\right), \boldsymbol{\beta}=\left(\beta_{1}, \ldots, \beta_{n-1}, 1\right)$ be in $\left(\mathbb{Q} \backslash \mathbb{Z}_{\leqslant 0}\right)^{n}$, let $p$ be a prime number such that $p>d_{\boldsymbol{\alpha}, \boldsymbol{\beta}}$ and $f(z):={ }_{n} F_{n-1}(\boldsymbol{\alpha}, \boldsymbol{\beta} ; z)$ belongs to $\mathbb{Z}_{(p)}[[z]]$. Suppose that $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ satisfies the $\boldsymbol{P}_{p, l}$ property, where $l$ is the order of $p$ in $\left(\mathbb{Z} / d_{\boldsymbol{\alpha}, \boldsymbol{\beta}} \mathbb{Z}\right)^{*}$. Then, for every $a \in\{1, \ldots, l\}$ and for every $r \in S_{\mathfrak{D}_{p}^{a-1}(\boldsymbol{\alpha}), \mathfrak{D}_{p}^{a-1}(\boldsymbol{\beta}), p}, f_{a, r} \in 1+z \mathbb{Z}_{(p)}[[z]]$ and

$$
f \equiv \sum_{r \in S_{\mathfrak{O}_{p}^{a-1}(\alpha), \mathfrak{D}_{p}^{a-1}(\mathcal{\beta}), p}} Q_{a, r}(z) f_{a, r}^{p^{a}} \bmod p,
$$

where, for every $r \in S_{\mathfrak{D}_{p}^{a-1}(\boldsymbol{\alpha}), \mathfrak{D}_{p}^{a-1}(\boldsymbol{\beta}), p}, Q_{a, r}(z)$ belongs to $\mathbb{Z}_{(p)}[z]$ and has degree less than $p^{a}$.

Section 7 is devoted to proving Lemma 6.1 and Lemma 6.2 will be proved in Section 8. The following remarks are useful in the proofs of Proposition 4.1 and Lemmas 6.2 and 9.1.

Remark 6.3. -
(1) If $\gamma \in \mathbb{Z}_{(p)}^{*}$ then $\mathfrak{D}_{p}(\gamma+1)=\mathfrak{D}_{p}(\gamma)$. Indeed, $p \mathfrak{D}_{p}(\gamma)-\gamma$ belongs to $\{1, \ldots, p-1\}$ because $\gamma \in \mathbb{Z}_{(p)}^{*}$. Whence, $p \mathfrak{D}_{p}(\gamma)-\gamma-1$ belongs to $\{0, \ldots, p-1\}$. So, $\mathfrak{D}_{p}(\gamma+1)=$ $\mathfrak{D}_{p}(\gamma)$.
(2) Let $p$ be a prime number and let $\boldsymbol{\alpha}=\left(\alpha_{1}, \ldots, \alpha_{n}\right), \boldsymbol{\beta}=\left(\beta_{1}, \ldots, \beta_{n-1}, 1\right)$ be in $\left(\mathbb{Z}_{(p)}\right)^{n}$. Suppose that $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ satisfies the $\mathbf{P}_{p, l}$ property. In this remark we show that, for all $(a, r) \in\{1, \ldots, l-1\} \times\{0, \ldots, p-1\}, \mathfrak{D}_{p}\left(\boldsymbol{\alpha}_{a, r}\right)=\mathfrak{D}_{p}^{a+1}(\boldsymbol{\alpha})$ and $\mathfrak{D}_{p}\left(\boldsymbol{\beta}_{a, r}\right)=$ $\mathfrak{D}_{p}^{a+1}(\boldsymbol{\beta})$. As $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ satisfies the $\mathbf{P}_{p, l}$ property and $a \in\{1, \ldots, l-1\}$ then, from
$(\mathbf{P} 1)$, we know that $\mathfrak{D}_{p}^{a}(\boldsymbol{\alpha}), \mathfrak{D}_{p}^{a}(\boldsymbol{\beta})$, belong to $\left(\mathbb{Z}_{(p)}^{*}\right)^{n}$. Let $w$ be in $\{1, \ldots, n\}$. If $w \in \mathcal{C}_{\mathfrak{D}_{p}^{a-1}(\boldsymbol{\alpha}), r}$ then $\alpha_{w, a, r}=\mathfrak{D}_{p}^{a}\left(\alpha_{w}\right)$. Thus, $\mathfrak{D}_{p}\left(\alpha_{w, a, r}\right)=\mathfrak{D}_{p}^{a+1}\left(\alpha_{w}\right)$. Now, if $w \in$ $\mathcal{P}_{\mathfrak{D}_{p}^{a-1}(\boldsymbol{\alpha}), r}$ then $\alpha_{w, a, r}=\mathfrak{D}_{p}^{a}\left(\alpha_{w}\right)+1$. Hence, by (1), $\mathfrak{D}_{p}\left(\mathfrak{D}_{p}^{a}\left(\alpha_{w}\right)+1\right)=\mathfrak{D}_{p}^{a+1}\left(\alpha_{w}\right)$ because $\mathfrak{D}_{p}^{a}\left(\alpha_{w}\right)$ belongs to $\mathbb{Z}_{(p)}^{*}$. Therefore, $\mathfrak{D}_{p}\left(\boldsymbol{\alpha}_{a, r}\right)=\mathfrak{D}_{p}^{a+1}(\boldsymbol{\alpha})$. In a similar fashion, it follows that $\mathfrak{D}_{p}\left(\boldsymbol{\beta}_{a, r}\right)=\mathfrak{D}_{p}^{a+1}(\boldsymbol{\beta})$.

## Remark 6.4. -

Let $p$ be a prime number and let $\boldsymbol{\alpha}=\left(\alpha_{1}, \ldots, \alpha_{n}\right), \boldsymbol{\beta}=\left(\beta_{1}, \ldots, \beta_{n-1}, 1\right)$ be in $\left(\mathbb{Z}_{(p)}\right)^{n}$. Suppose that $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ satisfies the $\mathbf{P}_{p, l}$ property. The goal of this remark is to show that, for all $(a, r) \in\{1, \ldots, l-1\} \times\{0, \ldots, p-1\},\left(\boldsymbol{\alpha}_{a, r}, \boldsymbol{\beta}_{a, r}\right)$ satisfies the $\mathbf{P}_{p, l^{\prime}}$ property, where $l^{\prime}$ is the order of $p$ in $\left(\mathbb{Z} / d_{\boldsymbol{\alpha}_{a, r}, \boldsymbol{\beta}_{a, r}} \mathbb{Z}\right)^{*}$. For this purpose, we will first show that, for any $1 \leqslant k \leqslant l$, there is $1 \leqslant s \leqslant l$ such that $\mathfrak{D}_{p}^{k}\left(\boldsymbol{\alpha}_{a, r}\right)=\mathfrak{D}_{p}^{s}(\boldsymbol{\alpha})$ and $\mathfrak{D}_{p}^{k}\left(\boldsymbol{\beta}_{a, r}\right)=\mathfrak{D}_{p}^{s}(\boldsymbol{\beta})$. From (2) of Remark 6.3, we have $\mathfrak{D}_{p}\left(\boldsymbol{\alpha}_{a, r}\right)=\mathfrak{D}_{p}^{a+1}(\boldsymbol{\alpha})$ and $\mathfrak{D}_{p}\left(\boldsymbol{\beta}_{a, r}\right)=\mathfrak{D}_{p}^{a+1}(\boldsymbol{\beta})$. Consequently, for all $k \in\{1, \ldots, l\}$, we get $\mathfrak{D}_{p}^{k}\left(\boldsymbol{\alpha}_{a, r}\right)=\mathfrak{D}_{p}^{a+k}(\boldsymbol{\alpha})$ and $\mathfrak{D}_{p}^{k}\left(\boldsymbol{\beta}_{a, r}\right)=\mathfrak{D}_{p}^{a+k}(\boldsymbol{\beta})$. Let $k$ be in $\{1, \ldots, l\}$ and let us write $a+k=s+t l$ with $0 \leqslant s<l$. We have $p^{l}=1 \bmod d_{\mathfrak{D}_{p}^{k}(\boldsymbol{\alpha}), \mathfrak{D}_{p}^{k}(\boldsymbol{\beta})}$ because, from the definition of $\mathfrak{D}_{p}$, it follows that $d_{\mathfrak{D}_{p}^{k}(\boldsymbol{\alpha}), \mathfrak{D}_{p}^{k}(\boldsymbol{\beta})}$ divides $d_{\boldsymbol{\alpha}, \boldsymbol{\beta}}{ }^{(2)}$ Further, from $(\mathbf{P} 1)$ we have $\mathfrak{D}_{p}^{k}(\boldsymbol{\alpha}), \mathfrak{D}_{p}^{k}(\boldsymbol{\beta}) \in \mathbb{Z}_{(p)}^{*} \cap(0,1]$. Then, by Lemma 2.1, we get that, for all $m \geqslant 1, \mathfrak{D}_{p}^{m l}\left(\mathfrak{D}_{p}^{k}(\boldsymbol{\alpha})\right)=\mathfrak{D}_{p}^{k}(\boldsymbol{\alpha})$ and $\mathfrak{D}_{p}^{m l}\left(\mathfrak{D}_{p}^{k}(\boldsymbol{\beta})\right)=\mathfrak{D}_{p}^{k}(\boldsymbol{\beta})$. So, $\mathfrak{D}_{p}^{k}\left(\boldsymbol{\alpha}_{a, r}\right)=\mathfrak{D}_{p}^{s}(\boldsymbol{\alpha})$ if $s \neq 0$ and if $s=0$, we have $\mathfrak{D}_{p}^{k}\left(\boldsymbol{\alpha}_{a, r}\right)=\mathfrak{D}_{p}^{t l}(\boldsymbol{\alpha})=\mathfrak{D}_{p}^{(t-1) l}\left(\mathfrak{D}_{p}^{l}(\boldsymbol{\alpha})\right)=\mathfrak{D}_{p}^{l}(\boldsymbol{\alpha})$. Similarly, we have $\mathfrak{D}_{p}^{k}\left(\boldsymbol{\beta}_{a, r}\right)=\mathfrak{D}_{p}^{s}(\boldsymbol{\beta})$ if $s \neq 0$ and $\mathfrak{D}_{p}^{k}\left(\boldsymbol{\beta}_{a, r}\right)=\mathfrak{D}_{p}^{l}(\boldsymbol{\beta})$ if $s=0$. Consequently, $\left(\boldsymbol{\alpha}_{a, r}, \boldsymbol{\beta}_{a, r}\right)$ satisfies the $\mathbf{P}_{p, l}$ property. Finally, from the definition of $\mathfrak{D}_{p}$ again, it immediately follows that $d_{\boldsymbol{\alpha}_{a, r}, \boldsymbol{\beta}_{a, r}}$ divides $d_{\boldsymbol{\alpha}, \boldsymbol{\beta}}$. Hence, if $l^{\prime}$ is the order of $p$ in $\left(\mathbb{Z} / d_{\boldsymbol{\alpha}_{a, r}, \boldsymbol{\beta}_{a, r}} \mathbb{Z}\right)^{*}$ then $l^{\prime}$ divides $l$ and therefore, $l^{\prime} \leqslant l$. So, $\left(\boldsymbol{\alpha}_{a, r}, \boldsymbol{\beta}_{a, r}\right)$ satisfies the $\mathbf{P}_{p, l^{\prime}}$ property.

Remark 6.5. - Let $\boldsymbol{\alpha}=\left(\alpha_{1}, \ldots, \alpha_{n}\right), \boldsymbol{\beta}=\left(\beta_{1}, \ldots, \beta_{n-1}, 1\right)$ be in $(\mathbb{Q} \cap(0,1])^{n}$ and let $p$ be a prime number such that $\boldsymbol{\alpha}, \boldsymbol{\beta}$ belong to $\left(\mathbb{Z}_{(p)}^{*}\right)^{n}$ and let $l$ be the order of $p$ in $\left(\mathbb{Z} / d_{\boldsymbol{\alpha}, \boldsymbol{\beta}} \mathbb{Z}\right)^{*}$.
(1) We show that, for all integers $m \geqslant 1$ and $r \in\{0, \ldots, p-1\}, \mathfrak{D}_{p}^{m}\left(\boldsymbol{\alpha}_{l, r}\right)=\mathfrak{D}_{p}^{m}(\boldsymbol{\alpha})$ and $\mathfrak{D}_{p}^{m}\left(\boldsymbol{\beta}_{l, r}\right)=\mathfrak{D}_{p}^{m}(\boldsymbol{\beta})$. Indeed, by assumption, $\boldsymbol{\alpha}, \boldsymbol{\beta}$ belong to $\left(\mathbb{Z}_{(p)}^{*}\right)^{n}$ and thus, according to Lemma 2.1, we get that $\mathfrak{D}_{p}^{l}(\boldsymbol{\alpha})=\boldsymbol{\alpha}$ and $\mathfrak{D}_{p}^{l}(\boldsymbol{\beta})=\boldsymbol{\beta}$. Therefore, for $1 \leqslant s \leqslant n, \alpha_{s, l, r}=\mathfrak{D}_{p}^{l}\left(\alpha_{s}\right)=\alpha_{s}$ if $s \in \mathcal{C}_{\mathfrak{D}_{p}^{l-1}(\boldsymbol{\alpha}), r}$ or $\alpha_{s, l, r}=\mathfrak{D}_{p}^{l}\left(\alpha_{s}\right)+1=$ $\alpha_{s}+1$ if $s \in \mathcal{P}_{\mathfrak{D}_{p}^{l-1}(\boldsymbol{\alpha}), r}$. From (1) of Remark 6.3, it follows that, for all $\gamma \in$ $\left\{\alpha_{1}, \ldots, \alpha_{n}, \beta_{1}, \ldots, \beta_{n}\right\}, \mathfrak{D}_{p}(\gamma+1)=\mathfrak{D}_{p}(\gamma)$. Consequently, for all integers $m \geqslant 1$, $\mathfrak{D}_{p}^{m}\left(\alpha_{s, l, r}\right)=\mathfrak{D}_{p}^{m}\left(\alpha_{s}\right)$ for all $1 \leqslant s \leqslant n$. In a similar fashion, one gets that, for all integers $m \geqslant 1, \mathfrak{D}_{p}^{m}\left(\beta_{s, l, r}\right)=\mathfrak{D}_{p}^{m}\left(\beta_{s}\right)$ for all $1 \leqslant s \leqslant n$. So that, for all $m \geqslant 1$, $\mathfrak{D}_{p}^{m}\left(\boldsymbol{\alpha}_{l, r}\right)=\mathfrak{D}_{p}^{m}(\boldsymbol{\alpha})$ and $\mathfrak{D}_{p}^{m}\left(\boldsymbol{\beta}_{l, r}\right)=\mathfrak{D}_{p}^{m}(\boldsymbol{\beta})$.
(2) As an immediately consequence of (1), we get that if $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ satisfies the $\mathbf{P}_{p, l}$ property then, for all $r \in\{0, \ldots, p-1\},\left(\boldsymbol{\alpha}_{l, r}, \boldsymbol{\beta}_{l, r}\right)$ satisfies also the $\mathbf{P}_{p, l}$ property. Furthermore, it is easily seen that $d_{\boldsymbol{\alpha}_{l, r}, \boldsymbol{\beta}_{l, r}}=d_{\boldsymbol{\alpha}, \boldsymbol{\beta}}$. Thus, $l$ is also the order of $p$ in $\left(\mathbb{Z} / d_{\boldsymbol{\alpha}_{l, r}, \boldsymbol{\beta}_{l, r}} \mathbb{Z}\right)^{*}$.

[^2]We can now prove Proposition 4.1.
Proof of Proposition 4.1. - Note that $f_{l, 0}$ is the hypergeometric series ${ }_{n} F_{n-1}(\boldsymbol{\alpha}, \boldsymbol{\beta} ; z)$ because, by assumption $\boldsymbol{\alpha}, \boldsymbol{\beta}$ belong to $\left(\mathbb{Z}_{(p)}^{*} \cap(0,1]\right)^{n}$ and thus, Lemma 2.1 implies $\mathfrak{D}_{p}^{l}(\boldsymbol{\alpha})=$ $\boldsymbol{\alpha}$ and $\mathfrak{D}_{p}^{l}(\boldsymbol{\beta})=\boldsymbol{\beta}$. We first prove that, for all $j \in S_{\mathfrak{D}_{p}^{l-1}(\boldsymbol{\alpha}), \mathfrak{D}_{p}^{l-1}(\boldsymbol{\beta}), p}, f_{l, j} \in 1+z \mathbb{Z}_{(p)}[[z]]$. By assumption, $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ satisfies the $\mathbf{P}_{p, l}$ property and $f_{l, 0}$ belongs to $\mathbb{Z}_{(p)}[[z]]$. By applying Lemma 6.2 to $f_{l, 0}(z)$ we get that, for every $j \in S_{\mathfrak{D}_{p}^{l-1}(\boldsymbol{\alpha}), \mathfrak{D}_{p}^{l-1}(\boldsymbol{\beta}), p}, f_{l, j} \in 1+z \mathbb{Z}_{(p)}[[z]]$.

Let $i$ be an arbitrary element in $S_{\mathfrak{D}_{p}^{l-1}(\boldsymbol{\alpha}), \mathfrak{D}_{p}^{l-1}(\boldsymbol{\beta}), p}$. We are going to prove that

$$
f_{l, i} \equiv \sum_{j \in S_{\mathcal{O}_{p}^{l-1}(\boldsymbol{\alpha}), \mathcal{D}_{p}^{l-1}(\boldsymbol{\beta}), p}} Q_{i, j} f_{l, j}^{p^{l}} \bmod p
$$

where, each $Q_{i, j}$ belongs to $\mathbb{Z}_{(p)}[z]$ with degree less than $p^{l}$. For this purpose, we are going to see that we can apply Lemma 6.2 to $f_{l, i}$. By definition $f_{l, i}$ is the hypergeometric series ${ }_{n} F_{n-1}\left(\boldsymbol{\alpha}_{l, i}, \boldsymbol{\beta}_{l, i} ; z\right)$. By (2) of Remark 6.5, we know that $l$ is also the order of $p$ in $\left(\mathbb{Z} / d_{\boldsymbol{\alpha}_{l, i}, \boldsymbol{\beta}_{l, i}} \mathbb{Z}\right)^{*}$ and that $\left(\boldsymbol{\alpha}_{l, i}, \boldsymbol{\beta}_{l, i}\right)$ satisfies the $\mathbf{P}_{p, l}$ property. Further, we also have $f_{l, i} \in \mathbb{Z}_{(p)}[[z]]$. So we are in a position to apply Lemma 6.2 to $f_{l, i}$ and therefore,

$$
\begin{equation*}
f_{l, i} \equiv \sum_{j \in S_{\mathfrak{O}_{p}^{l-1}\left(\boldsymbol{\alpha}_{l, i}\right), \mathfrak{D}_{p}^{l-1}\left(\boldsymbol{\beta}_{l, i}\right), p}} Q_{i, j} g_{i, j}^{p^{l}} \bmod p \tag{6.1}
\end{equation*}
$$

where, $Q_{i, j}(z) \in \mathbb{Z}_{(p)}[z]$ has degree less than $p^{l}$ and

$$
g_{i, j}=\sum_{m \geqslant 0}\left(\frac{\prod_{s \in \mathcal{C}_{\mathcal{D}_{p}^{l-1}\left(\alpha_{l, i}\right), j}}\left(\mathfrak{D}_{p}^{l}\left(\alpha_{s, l, i}\right)\right)_{m} \prod_{s \in \mathcal{P}_{\mathcal{P}_{p}^{l-1}\left(\boldsymbol{\alpha}_{l, i}\right), j}}\left(\mathfrak{D}_{p}^{l}\left(\alpha_{s, l, i}\right)+1\right)_{m}}{\prod_{s \in \mathcal{C}_{\mathcal{O}_{p}^{l-1}\left(\boldsymbol{\beta}_{l, i}\right), j}}\left(\mathfrak{D}_{p}^{l}\left(\beta_{s, l, i}\right)\right)_{m} \prod_{s \in \mathcal{P}_{\mathcal{D}_{p}^{l-1}\left(\boldsymbol{\beta}_{l, i}\right), j}}\left(\mathfrak{D}_{p}^{l}\left(\beta_{s, l, i}\right)+1\right)_{m}}\right) z^{m} \in \mathbb{Z}_{(p)}[[z]] .
$$

We have already seen that $\mathfrak{D}_{p}^{l}(\boldsymbol{\alpha})=\boldsymbol{\alpha}$. Thus, $\alpha_{s, l, i}=\alpha_{s}$ if $s \in \mathcal{C}_{\mathfrak{D}_{p}^{l-1}\left(\boldsymbol{\alpha}_{l, i}\right), j}$ and $\alpha_{s, l, i}=\alpha_{s}+1$ if $s \in \mathcal{P}_{\mathfrak{D}_{p}^{l-1}\left(\boldsymbol{\alpha}_{l, i}\right), j}$. Since, by assumption $\boldsymbol{\alpha}$ belongs to $\left(\mathbb{Z}_{(p)}^{*} \cap(0,1]\right)^{n}$, by (1) of Remark 6.5, we deduce that $\mathfrak{D}_{p}^{l}\left(\alpha_{s, l, i}\right)=\mathfrak{D}_{p}^{l}\left(\alpha_{s}\right)=\alpha_{s}$. In a similar way, one obtains $\mathfrak{D}_{p}^{l}\left(\beta_{s, l, i}\right)=\mathfrak{D}_{p}^{l}\left(\beta_{s}\right)=$ $\beta_{s}$. Hence,

$$
g_{i, j}=\sum_{m \geqslant 0}\left(\frac{\prod_{s \in \mathcal{C}_{\mathcal{D}_{p}^{l-1}\left(\boldsymbol{\alpha}_{l, i}\right), j}}\left(\alpha_{s}\right)_{m} \prod_{s \in \mathcal{P}_{\mathfrak{D}_{p}^{l-1}\left(\boldsymbol{\alpha}_{l, i}\right), j}}\left(\alpha_{s}+1\right)_{m}}{\prod_{s \in \mathcal{C}_{\mathfrak{O}_{p}^{l-1}\left(\boldsymbol{\beta}_{l, i}\right), j}}\left(\beta_{s}\right)_{m} \prod_{s \in \mathcal{P}_{\mathfrak{D}_{p}^{l-1}\left(\boldsymbol{\beta}_{l, i}\right), j}}\left(\beta_{s}+1\right)_{m}}\right) z^{m} .
$$

Suppose that $l \geqslant 2$. We want to see that $g_{i, j}=f_{l, j}$. Since $l \geqslant 2$, by (1) of Remark 6.5, we know that $\mathfrak{D}_{p}^{l-1}\left(\boldsymbol{\alpha}_{l, i}\right)=\mathfrak{D}_{p}^{l-1}(\boldsymbol{\alpha})$ and that $\mathfrak{D}_{p}^{l-1}\left(\boldsymbol{\beta}_{l, i}\right)=\mathfrak{D}_{p}^{l-1}(\boldsymbol{\beta})$. Therefore, we have the equality, $S_{\mathfrak{D}_{p}^{l-1}\left(\boldsymbol{\alpha}_{l, i}\right), \mathfrak{D}_{p}^{l-1}\left(\boldsymbol{\beta}_{l, i}\right), p}=S_{\mathfrak{D}_{p}^{l-1}(\boldsymbol{\alpha}), \mathfrak{D}_{p}^{l-1}(\boldsymbol{\beta}), p}$. Whence, $g_{i, j}=f_{l, j}$ for all $j \in$ $S_{\mathfrak{D}_{p}^{l-1}\left(\boldsymbol{\alpha}_{l, i}\right), \mathfrak{D}_{p}^{l-1}\left(\boldsymbol{\beta}_{l, i}\right), p}$. So, from Equation (6.1), we get

$$
f_{l, i} \equiv \sum_{j \in S_{\mathfrak{O}_{p}^{l-1}(\boldsymbol{\alpha}), \mathfrak{D}_{p}^{l-1}(\boldsymbol{\beta}), p}} Q_{i, j} f_{l, j}^{p^{l}} \bmod p
$$

Suppose now that $l=1$. We want to see that $g_{i, j}=f_{1, \sigma(j)}$, where $\sigma: S_{\boldsymbol{\alpha}_{l, i}, \boldsymbol{\beta}_{l, i}, p} \rightarrow$ $S_{\boldsymbol{\alpha}, \boldsymbol{\beta}, p}$ is the map given by Lemma 6.1. Since $l=1$, it is clear that $S_{\mathfrak{D}_{p}^{l-1}\left(\boldsymbol{\alpha}_{l, i}\right), \mathfrak{D}_{p}^{l-1}\left(\boldsymbol{\beta}_{l, i}\right), p}=$
$S_{\boldsymbol{\alpha}_{l, i}, \boldsymbol{\beta}_{l, i}, p}$. Since $l$ is the order of $p$ in $\left(\mathbb{Z} / d_{\boldsymbol{\alpha}, \boldsymbol{\beta}} \mathbb{Z}\right)^{*}$ and by hypotheses, $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ belong to $\left(\mathbb{Z}_{(p)}^{*}\right)^{n}$, by using Lemma 2.1, we obtain $\mathfrak{D}_{p}(\boldsymbol{\alpha})=\boldsymbol{\alpha}$ and $\mathfrak{D}_{p}(\boldsymbol{\beta})=\boldsymbol{\beta}$. Further, $p$ does not divide $d_{\boldsymbol{\alpha}, \boldsymbol{\beta}}$ because, by assumption, for all $i, j \in\{1, \ldots, n\}, \alpha_{i}, \beta_{j}$ belong to $\mathbb{Z}_{(p)}^{*}$. We also have, by assumption again, $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ satisfies the $\mathbf{P}_{p, 1}$ property. So, by B) of Lemma 6.1, we infer that, for all $j \in S_{\boldsymbol{\alpha}_{l, i}, \boldsymbol{\beta}_{l, i}, p}$, we have $\mathcal{P}_{\boldsymbol{\alpha}_{l, i}, j}=\mathcal{P}_{\boldsymbol{\alpha}, \sigma(j)}, \mathcal{C}_{\boldsymbol{\alpha}_{l, i}, j}=\mathcal{C}_{\boldsymbol{\alpha}, \sigma(j)}, \mathcal{P}_{\boldsymbol{\alpha}_{l, i}, j}=\mathcal{P}_{\boldsymbol{\alpha}, \sigma(j)}$, and $\mathcal{C}_{\boldsymbol{\alpha}_{l, i}, j}=\mathcal{C}_{\boldsymbol{\alpha}, \sigma(j)}$. Thus, for all $j \in S_{\boldsymbol{\alpha}_{l, i}, \boldsymbol{\beta}_{l, i}, p}, g_{i, j}=f_{1, \sigma(j)}$. Since, by A) of Lemma 6.1, $\sigma: S_{\boldsymbol{\alpha}_{l, i}, \boldsymbol{\beta}_{l, i}, p} \rightarrow S_{\boldsymbol{\alpha}, \boldsymbol{\beta}, p}$ is a bijective map, it follows from Equation (6.1) that

$$
f_{1, i} \equiv \sum_{j \in S_{\boldsymbol{\alpha}, \boldsymbol{\beta}, p}} Q_{i, \tau(j)} f_{1, j}^{p} \bmod p
$$

This completes the proof because $i$ is an arbitrary element in $S_{\mathfrak{D}_{p}^{l-1}(\boldsymbol{\alpha}), \mathfrak{D}_{p}^{l-1}(\boldsymbol{\beta}), p}$.

## 7. Proof of Lemma 6.1

A) Let $t$ be in $S_{\boldsymbol{\alpha}_{a, r}, \boldsymbol{\beta}_{a, r}, p}$. Then $t \bmod p \equiv 1-\beta_{s, a, r} \bmod p$ for some $s \in\{1, \ldots, n\}$ and $v_{p}\left(\mathcal{Q}_{\boldsymbol{\alpha}_{a, r}, \boldsymbol{\beta}_{a, r}}(t)\right)=0$. We are going to see that $\sigma$ is well-defined. For this purpose, we first show that if there exists $s^{\prime} \in\{1, \ldots, n\}$ such that $t \bmod p \equiv 1-\beta_{s^{\prime}, a, r} \bmod p$ then $s \in \mathcal{C}_{\mathfrak{D}_{p}^{a-1}(\boldsymbol{\beta}), r}$ if and only if $s^{\prime} \in \mathcal{C}_{\mathfrak{D}_{p}^{a-1}(\boldsymbol{\beta}), r}$ and second, we prove that $\sigma(t) \in S_{\mathfrak{D}_{p}^{a}(\boldsymbol{\alpha}), \mathfrak{D}_{p}^{a}(\boldsymbol{\beta}), p}$. Suppose that $s \in \mathcal{C}_{\mathfrak{D}_{p}^{a-1}(\boldsymbol{\beta}), r}$. Assume for contradiction that $s^{\prime} \in \mathcal{P}_{\mathfrak{D}_{p}^{a-1}(\boldsymbol{\beta}), r}$. Thus, $\beta_{s, a, r}=$ $\mathfrak{D}_{p}^{a}\left(\beta_{s}\right)$ and $\beta_{s^{\prime}, a, r}=\mathfrak{D}_{p}^{a}\left(\beta_{s^{\prime}}\right)+1$. Then $\mathfrak{D}_{p}^{a}\left(\beta_{s}\right) \equiv \mathfrak{D}_{p}^{a}\left(\beta_{s^{\prime}}\right)+1 \bmod p$ because $1-\beta_{s, a, r} \bmod$ $p \equiv t \bmod p \equiv 1-\beta_{s^{\prime}, a, r} \bmod p$. Hence, $1+\mathfrak{D}_{p}^{a}\left(\beta_{s^{\prime}}\right)-\mathfrak{D}_{p}^{a}\left(\beta_{s}\right) \in p \mathbb{Z}_{p}$. That is a contradiction to ( $\mathbf{P} 5$ ). Therefore, we have $s^{\prime} \in \mathcal{C}_{\mathfrak{D}_{p}^{a-1}(\boldsymbol{\beta}), r}$. In a similar way, one shows that if $s^{\prime} \in \mathcal{C}_{\mathfrak{D}_{p}^{a-1}(\boldsymbol{\beta}), r}$ then $s \in \mathcal{C}_{\mathfrak{D}_{p}^{a-1}(\boldsymbol{\beta}), r}$.

We now prove that $\sigma(t) \in S_{\mathfrak{D}_{p}^{a}(\boldsymbol{\alpha}), \mathfrak{D}_{p}^{a}(\boldsymbol{\beta}), p}$. By definition, $t \bmod p \equiv 1-\beta_{s, a, r} \bmod p$ for some $s \in\{1, \ldots, n\}$. Thus

$$
t \bmod p=\left\{\begin{array}{lll}
1-\mathfrak{D}_{p}^{a}\left(\beta_{s}\right) \bmod p & \text { if } & s \in \mathcal{C}_{\mathfrak{D}_{p}^{a-1}(\boldsymbol{\beta}), r} \\
-\mathfrak{D}_{p}^{a}\left(\beta_{s}\right) \bmod p & \text { if } & s \in \mathcal{P}_{\mathfrak{D}_{p}^{a-1}(\boldsymbol{\beta}), r}
\end{array}\right.
$$

- Suppose that $s \in \mathcal{C}_{\mathfrak{D}_{p}^{a-1}(\boldsymbol{\beta}), r}$. Then, $t \bmod p=1-\mathfrak{D}_{p}^{a}\left(\beta_{s}\right) \bmod p$ and therefore, $t \in$ $E_{\mathfrak{D}_{p}^{a}(\boldsymbol{\alpha}), \mathfrak{D}_{p}^{a}(\boldsymbol{\beta}), p}$. We now show that $t \in S_{\mathfrak{D}_{p}^{a}(\boldsymbol{\alpha}), \mathfrak{D}_{p}^{a}(\boldsymbol{\beta}), p}$. It is clear that we have the following equality

$$
\begin{equation*}
\mathcal{Q}_{\boldsymbol{\alpha}_{a, r}, \boldsymbol{\beta}_{a, r}}(t)=\mathcal{Q}_{\mathfrak{D}_{p}^{a}(\boldsymbol{\alpha}), \mathfrak{D}_{p}^{a}(\boldsymbol{\beta})}(t) \cdot \frac{\prod_{w \in \mathcal{P}_{\mathfrak{D}_{p}^{a-1}(\boldsymbol{\alpha}), r}}\left(\mathfrak{D}_{p}^{a}\left(\alpha_{w}\right)+t\right)}{\prod_{w \in \mathcal{P}_{\mathfrak{O}_{p}^{a-1}(\boldsymbol{\beta}), r}}\left(\mathfrak{D}_{p}^{a}\left(\beta_{w}\right)+t\right)} \cdot \frac{\prod_{w \in \mathcal{P}_{\mathfrak{O}_{p}^{a-1}(\boldsymbol{\beta}), r}} \mathfrak{D}_{p}^{a}\left(\beta_{w}\right)}{\prod_{w \in \mathcal{P}_{\mathfrak{O}_{p}^{a-1}(\boldsymbol{\alpha}), r}^{a}} \mathfrak{D}_{p}^{a}\left(\alpha_{w}\right)} . \tag{7.1}
\end{equation*}
$$

By $(\mathbf{P} 1)$, we know that $\mathfrak{D}_{p}^{a}\left(\beta_{1}\right), \ldots, \mathfrak{D}_{p}^{a}\left(\beta_{n}\right), \mathfrak{D}_{p}^{a}\left(\alpha_{1}\right), \ldots, \mathfrak{D}_{p}^{a}\left(\alpha_{n}\right)$ belong to $\mathbb{Z}_{(p)}^{*}$. Then,

$$
\left(\prod_{w \in \mathcal{P}_{\mathfrak{O}_{p}^{a-1}(\mathcal{\beta}), r}} \mathfrak{D}_{p}^{a}\left(\beta_{w}\right)\right) /\left(\prod_{w \in \mathcal{P}_{\mathfrak{O}_{p}^{a-1}(\alpha), r}} \mathfrak{D}_{p}^{a}\left(\alpha_{w}\right)\right) \in \mathbb{Z}_{(p)}^{*}
$$

Now, assume for contradiction that there is $\gamma \in\left\{\alpha_{1}, \ldots, \alpha_{n}, \beta_{1}, \ldots, \beta_{n}\right\}$ such that $\mathfrak{D}_{p}^{a}(\gamma)+t$ belongs to $p \mathbb{Z}_{(p)}$. Then $t=p \mathfrak{D}_{p}^{a+1}(\gamma)-\mathfrak{D}_{p}^{a}(\gamma)$ because $0 \leqslant t<p$. As $t \bmod p=1-$ $\mathfrak{D}_{p}^{a}\left(\beta_{s}\right) \bmod p$ then $1-\mathfrak{D}_{p}^{a}\left(\beta_{s}\right)+\mathfrak{D}_{p}^{a}(\gamma) \in p \mathbb{Z}_{(p)}$, which is a contradiction to (P5). Consequently, $\mathfrak{D}_{p}^{a}\left(\alpha_{1}\right)+t, \ldots, \mathfrak{D}_{p}^{a}\left(\alpha_{n}\right)+t, \mathfrak{D}_{p}^{a}\left(\beta_{1}\right)+t, \ldots, \mathfrak{D}_{p}^{a}\left(\beta_{n}\right)+t$ belong to $\mathbb{Z}_{(p)}^{*}$. Therefore,

$$
\left(\prod_{w \in \mathcal{P}_{\mathfrak{O}_{p}^{a-1}(\alpha), r}}\left(\mathfrak{D}_{p}^{a}\left(\alpha_{w}\right)+t\right)\right) /\left(\prod_{w \in \mathcal{P}_{\mathfrak{O}_{p}^{a-1}(\boldsymbol{\beta}), r}}\left(\mathfrak{D}_{p}^{a}\left(\beta_{w}\right)+t\right)\right) \in \mathbb{Z}_{(p)}^{*}
$$

Thus, from (7.1), we get $v_{p}\left(\mathcal{Q}_{\mathfrak{D}_{p}^{a}(\boldsymbol{\alpha}), \mathfrak{D}_{p}^{a}(\boldsymbol{\beta})}(t)\right)=0$ because $v_{p}\left(\mathcal{Q}_{\boldsymbol{\alpha}_{a, r}, \boldsymbol{\beta}_{a, r}}(t)\right)=0$. So that $t \in S_{\mathfrak{D}_{p}^{a}(\boldsymbol{\alpha}), \mathfrak{D}_{p}^{a}(\boldsymbol{\beta}), p}$.

- Suppose that $s \in \mathcal{P}_{\mathfrak{D}_{p}^{a-1}(\boldsymbol{\beta}), r}$. Then, $t \bmod p \equiv-\mathfrak{D}_{p}^{a}\left(\beta_{s}\right) \bmod p$.

We first prove that $t+1 \in E_{\mathfrak{D}_{p}^{a}(\boldsymbol{\alpha}), \mathfrak{D}_{p}^{a}(\boldsymbol{\beta}), p}$. As $0 \leqslant r<p$ then, (1) $)_{r} \notin p \mathbb{Z}_{(p)}$. By assumption, $s \in \mathcal{P}_{\mathfrak{D}_{p}^{a-1}(\boldsymbol{\beta}), r}$ and thus, $\left(\mathfrak{D}_{p}^{a-1}\left(\beta_{s}\right)\right)_{r} \in p \mathbb{Z}_{(p)}$. Hence, $\beta_{s} \neq 1$. Now, assume for contradiction that $t=p-1$. Since $t \bmod p=-\mathfrak{D}_{p}^{a}\left(\beta_{j}\right) \bmod p$, we have $p-1+\mathfrak{D}_{p}^{a}\left(\beta_{s}\right) \in p \mathbb{Z}_{(p)}$. Therefore, we have $p \mathfrak{D}_{p}^{a+1}\left(\beta_{s}\right)-\mathfrak{D}_{p}^{a}\left(\beta_{s}\right)=p-1$. Since $\beta_{s} \neq 1$, it follows that $p-1 \in I_{\boldsymbol{\beta}}^{(a+1)}$. But, according to ( $\mathbf{P} 4$ ), $p-1$ does not belong to $I_{\boldsymbol{\beta}}^{(a+1)}$. For this reason $t \neq p-1$. As $S_{\boldsymbol{\alpha}_{a, r}, \boldsymbol{\beta}_{a, r}, p} \subset$ $\{0,1, \ldots, p-1\}$ then $t<p-1$. Hence, $t+1 \in E_{\mathfrak{D}_{p}^{a}(\boldsymbol{\alpha}), \mathfrak{D}_{p}^{a}(\boldsymbol{\beta}), p}$ because $t+1 \leqslant p-1$ and $t+1 \bmod p \equiv 1-\mathfrak{D}_{p}^{a}\left(\beta_{s}\right) \bmod p$.

We now proceed to see that $t+1 \in S_{\mathfrak{D}_{p}^{a}(\boldsymbol{\alpha}), \mathfrak{D}_{p}^{a}(\boldsymbol{\beta}), p}$. It is clear that we have the following equality

$$
\begin{equation*}
\mathcal{Q}_{\mathfrak{D}_{p}^{a}(\boldsymbol{\alpha}), \mathfrak{D}_{p}^{a}(\boldsymbol{\beta})}(t+1)=\mathcal{Q}_{\boldsymbol{\alpha}_{a, r}, \boldsymbol{\beta}_{a, r}}(t) \cdot \frac{\prod_{w \in \mathcal{C}_{\mathfrak{O}_{p}^{a-1}(\boldsymbol{\alpha}), r}}\left(\mathfrak{D}_{p}^{a}\left(\alpha_{w}\right)+t\right)}{\prod_{w \in \mathcal{C}_{\mathcal{D}_{p}^{a-1}(\boldsymbol{\beta}), r}}\left(\mathfrak{D}_{p}^{a}\left(\beta_{w}\right)+t\right)} \cdot \frac{\prod_{w \in \mathcal{P}_{\mathfrak{O}_{p}^{a-1}(\alpha), r}} \mathfrak{D}_{p}^{a}\left(\alpha_{w}\right)}{\prod_{w \in \mathcal{P}_{\mathcal{O}_{p}^{a-1}(\boldsymbol{\beta}), r}} \mathfrak{D}_{p}^{a}\left(\beta_{w}\right)} . \tag{7.2}
\end{equation*}
$$

It follows from $(\mathbf{P} 1)$ that $\mathfrak{D}_{p}^{a}\left(\alpha_{1}\right), \ldots, \mathfrak{D}_{p}^{a}\left(\alpha_{n}\right), \mathfrak{D}_{p}^{a}\left(\beta_{1}\right), \ldots, \mathfrak{D}_{p}^{a}\left(\beta_{n}\right)$ belong to $\mathbb{Z}_{(p)}^{*}$. Then,

$$
\left(\prod_{w \in \mathcal{P}_{\mathfrak{O}_{p}^{a-1}(\alpha), r}} \mathfrak{D}_{p}^{a}\left(\alpha_{w}\right)\right) /\left(\prod_{w \in \mathcal{P}_{\mathfrak{O}_{p}^{a-1}(\boldsymbol{\beta}), r}} \mathfrak{D}_{p}^{a}\left(\beta_{w}\right)\right) \in \mathbb{Z}_{(p)}^{*}
$$

Now, assume for contradiction that there is $\gamma \in\left\{\alpha_{1}, \ldots, \alpha_{n}\right\}$ such that $\mathfrak{D}_{p}^{a}(\gamma)+t$ belongs to $p \mathbb{Z}_{(p)}$. Since $t \bmod p=-\mathfrak{D}_{p}^{a}\left(\beta_{s}\right) \bmod p$, it follows that $\mathfrak{D}_{p}^{a}(\gamma)-\mathfrak{D}_{p}^{a}\left(\beta_{s}\right) \in p \mathbb{Z}_{(p)}$. That is a contradiction because, according to (P2), $\mathfrak{D}_{p}^{a}(\gamma)-\mathfrak{D}_{p}^{a}\left(\beta_{s}\right) \notin p \mathbb{Z}_{(p)}$. For this reason, the elements $\mathfrak{D}_{p}^{a}\left(\alpha_{1}\right)+t, \ldots, \mathfrak{D}_{p}^{a}\left(\alpha_{n}\right)+t$ belong to $\mathbb{Z}_{(p)}^{*}$.

Again, suppose, to derive a contradiction, that there is $w \in \mathcal{C}_{\mathfrak{D}_{p}^{a-1}(\boldsymbol{\beta}), r}$ such that $\mathfrak{D}_{p}^{a}\left(\beta_{w}\right)+$ $t$ belongs to $p \mathbb{Z}_{(p)}$. Since $t \equiv-\mathfrak{D}_{p}^{a}\left(\beta_{s}\right) \bmod p$, we obtain $\mathfrak{D}_{p}^{a}\left(\beta_{w}\right)-\mathfrak{D}_{p}^{a}\left(\beta_{s}\right) \in p \mathbb{Z}_{(p)}$. Then, according to (P3), $\beta_{w}=\beta_{s}$. On the one hand, we have $\left(\mathfrak{D}_{p}^{a-1}\left(\beta_{s}\right)\right)_{r} \in p \mathbb{Z}_{(p)}$ because $s \in$ $\mathcal{P}_{\mathfrak{D}_{p}^{a-1}(\boldsymbol{\beta}), r}$. On the other hand, $\left(\mathfrak{D}_{p}^{a-1}\left(\beta_{s}\right)\right)_{r} \notin p \mathbb{Z}_{(p)}$ because $\beta_{w}=\beta_{s}$ and $w \in \mathcal{C}_{\mathfrak{D}_{p}^{a-1}(\boldsymbol{\beta}), r}$. So that, we have a contradiction. For this reason, for every $w \in \mathcal{C}_{\mathfrak{D}_{p}^{a-1}(\boldsymbol{\beta}), r}, \mathfrak{D}_{p}^{a}\left(\beta_{w}\right)+t$ belongs

## ALGEBRAICITY MODULO P OF GENERALIZED HYPERGEOMETRIC SERIES ${ }_{n} F_{n-1}$

to $\mathbb{Z}_{(p)}^{*}$. Consequently, the element

$$
\left(\prod_{w \in \mathcal{C}_{\mathfrak{O}_{p}^{a-1}(\boldsymbol{\alpha}), r}}\left(\mathfrak{D}_{p}^{a}\left(\alpha_{w}\right)+t\right)\right) /\left(\prod_{w \in \mathcal{C}_{\mathbb{O}_{p}^{a-1}(\boldsymbol{\beta}), r}}\left(\mathfrak{D}_{p}^{a}\left(\beta_{w}\right)+t\right)\right) \in \mathbb{Z}_{(p)}^{*}
$$

Then, it follows from Equation (7.2) that $v_{p}\left(\mathcal{Q}_{\mathfrak{D}_{p}^{a}(\boldsymbol{\alpha}), \mathfrak{D}_{p}^{a}(\boldsymbol{\beta})}(t+1)\right)=0$ because $v_{p}\left(\mathcal{Q}_{\boldsymbol{\alpha}_{a, r}, \boldsymbol{\beta}_{a, r}}(t)\right)=$ 0 . So that, we have $t+1 \in S_{\mathfrak{D}_{p}^{a}(\boldsymbol{\alpha}), \mathfrak{D}_{p}^{a}(\boldsymbol{\beta}), p}$.

Therefore, we have $\sigma$ is well-defined. In order to prove that $\sigma$ is a bijective map we are going to show that its inverse is $\tau: S_{\mathfrak{D}_{p}^{a}(\boldsymbol{\alpha}), \mathfrak{D}_{p}^{a}(\boldsymbol{\beta}), p} \rightarrow S_{\boldsymbol{\alpha}_{a, r}, \boldsymbol{\beta}_{a, r}, p}$.

Let $t$ be in $S_{\mathfrak{D}_{p}^{a}(\boldsymbol{\alpha}), \mathfrak{D}_{p}^{a}(\boldsymbol{\beta}), p}$. Then $v_{p}\left(\mathcal{Q}_{\mathfrak{D}_{p}^{a}(\boldsymbol{\alpha}), \mathfrak{D}_{p}^{a}(\boldsymbol{\beta})}(t)\right)=0$ and $t \bmod p \equiv 1-\mathfrak{D}_{p}^{a}\left(\beta_{s}\right) \bmod p$ for some $s \in\{1, \ldots, n\}$. We are going to see that $\tau$ is well-defined. For this purpose, we first show that if there exists $s^{\prime} \in\{1, \ldots, n\}$ such that $t \bmod p \equiv 1-\mathfrak{D}_{p}^{a}\left(\beta_{s^{\prime}}\right) \bmod p$ then, $s \in \mathcal{C}_{\mathfrak{D}_{p}^{a-1}(\boldsymbol{\beta}), r}$ if and only if $s^{\prime} \in \mathcal{C}_{\mathfrak{D}_{p}^{a-1}(\boldsymbol{\beta}), r}$ and second, we prove that $\tau(t) \in S_{\boldsymbol{\alpha}_{a, r}, \boldsymbol{\beta}_{a, r}, p}$. Suppose that $s \in \mathcal{C}_{\mathfrak{D}_{p}^{a-1}(\boldsymbol{\beta}), r}$. Note that $\mathfrak{D}_{p}^{a}\left(\beta_{s}\right)-\mathfrak{D}_{p}^{a}\left(\beta_{s^{\prime}}\right) \in p \mathbb{Z}_{(p)}$ because 1- $\mathfrak{D}_{p}^{a}\left(\beta_{s}\right) \bmod p=$ $t \bmod p=1-\mathfrak{D}_{p}^{a}\left(\beta_{s}^{\prime}\right)$. So according to $(\mathbf{P} 3), \beta_{s}=\beta_{s^{\prime}}$. So, $\left(\mathfrak{D}_{p}^{a-1}\left(\beta_{s^{\prime}}\right)\right)_{r} \notin p \mathbb{Z}_{(p)}$ because $s \in \mathcal{C}_{\mathfrak{D}_{p}^{a-1}(\boldsymbol{\beta}), r}$. Hence, $s^{\prime} \in \mathcal{C}_{\mathfrak{D}_{p}^{a-1}(\boldsymbol{\beta}), r}$. In a similar way, one shows that if $s^{\prime} \in \mathcal{C}_{\mathfrak{D}_{p}^{a-1}(\boldsymbol{\beta}), r}$ then $s \in \mathcal{C}_{\mathfrak{D}_{p}^{a-1}(\boldsymbol{\beta}), r}$. We now proceed to show that $\tau(t) \in S_{\boldsymbol{\alpha}_{a, r}, \boldsymbol{\beta}_{a, r}, p}$.

- Suppose that $t \bmod p \equiv 1-\mathfrak{D}_{p}^{a}\left(\beta_{s}\right) \bmod p$ with $s \in \mathcal{C}_{\mathfrak{D}_{p}^{a-1}(\boldsymbol{\beta}), r}$. Then $\beta_{s, a, r}=\mathfrak{D}_{p}^{a}\left(\beta_{s}\right)$ and therefore, $t \in E_{\boldsymbol{\alpha}_{a, r}, \boldsymbol{\beta}_{a, r}, p}$. We want to see that $t \in S_{\boldsymbol{\alpha}_{a, r}, \boldsymbol{\beta}_{a, r}, p}$. We have the following equality

$$
\begin{equation*}
\mathcal{Q}_{\mathfrak{D}_{p}^{a}(\boldsymbol{\alpha}), \mathfrak{D}_{p}^{a}(\boldsymbol{\beta})}(t)=\mathcal{Q}_{\boldsymbol{\alpha}_{a, r}, \boldsymbol{\beta}_{a, r}}(t) \cdot \frac{\prod_{w \in \mathcal{P}_{\mathfrak{O}_{p}^{a-1}(\boldsymbol{\beta}), r}}\left(\mathfrak{D}_{p}^{a}\left(\beta_{w}\right)+t\right)}{\prod_{w \in \mathcal{P}_{\mathfrak{D}_{p}^{a-1}(\boldsymbol{\alpha}), r}^{a}}\left(\mathfrak{D}_{p}^{a}\left(\alpha_{w}\right)+t\right)} \cdot \frac{\prod_{w \in \mathcal{P}_{\mathfrak{O}_{p}^{a-1}(\boldsymbol{\alpha}), r}} \mathfrak{D}_{p}^{a}\left(\alpha_{w}\right)}{\prod_{w \in \mathcal{P}_{\mathfrak{O}_{p}^{a-1}(\boldsymbol{\beta}), r}^{a}} \mathfrak{D}_{p}^{a}\left(\beta_{w}\right)} . \tag{7.3}
\end{equation*}
$$

By $(\mathbf{P} 1)$, we know that $\mathfrak{D}_{p}^{a}\left(\alpha_{1}\right), \ldots, \mathfrak{D}_{p}^{a}\left(\alpha_{n}\right), \mathfrak{D}_{p}^{a}\left(\beta_{1}\right), \ldots, \mathfrak{D}_{p}^{a}\left(\beta_{n}\right)$ belong to $\mathbb{Z}_{(p)}^{*}$. Therefore,

$$
\left(\prod_{w \in \mathcal{P}_{\mathfrak{O}_{p}^{a-1}(\alpha), r}} \mathfrak{D}_{p}^{a}\left(\alpha_{w}\right)\right) /\left(\prod_{w \in \mathcal{P}_{\mathfrak{O}_{p}^{a-1}(\mathcal{\beta}), r}} \mathfrak{D}_{p}^{a}\left(\beta_{w}\right)\right) \in \mathbb{Z}_{(p)}^{*}
$$

Now, assume for contradiction that there is $\gamma \in\left\{\alpha_{1}, \ldots, \alpha_{n}, \beta_{1}, \ldots, \beta_{n}\right\}$ such that $\mathfrak{D}_{p}^{a}(\gamma)+t$ belongs to $p \mathbb{Z}_{(p)}$. Since $0 \leqslant t<p$, it follows that $t=p \mathfrak{D}_{p}^{a+1}(\gamma)-\mathfrak{D}_{p}^{a}(\gamma)$. As $t \bmod p \equiv 1-$ $\mathfrak{D}_{p}^{a}\left(\beta_{s}\right) \bmod p$ then $1-\mathfrak{D}_{p}^{a}\left(\beta_{s}\right)+\mathfrak{D}_{p}^{a}(\gamma) \in p \mathbb{Z}_{(p)}$. This a contradiction to ( $\left.\mathbf{P} 5\right)$. Consequently, the elements $\mathfrak{D}_{p}^{a}\left(\beta_{1}\right)+t, \ldots, \mathfrak{D}_{p}^{a}\left(\beta_{n}\right)+t, \mathfrak{D}_{p}^{a}\left(\alpha_{1}\right)+t, \ldots, \mathfrak{D}_{p}^{a}\left(\alpha_{n}\right)+t$ belong to $\mathbb{Z}_{(p)}^{*}$. Thus,

$$
\left(\prod_{w \in \mathcal{P}_{\mathfrak{O}_{p}^{a-1}(\boldsymbol{\beta}), r}}\left(\mathfrak{D}_{p}^{a}\left(\beta_{w}\right)+t\right)\right) /\left(\prod_{w \in \mathcal{P}_{\mathfrak{O}_{p}^{a-1}(\alpha), r}}\left(\mathfrak{D}_{p}^{a}\left(\alpha_{w}\right)+t\right)\right) \in \mathbb{Z}_{(p)}^{*}
$$

Then, from Equation (7.3), we have $v_{p}\left(\mathcal{Q}_{\boldsymbol{\alpha}_{a, r}, \boldsymbol{\beta}_{a, r}}(t)\right)=0$ because $v_{p}\left(\mathcal{Q}_{\mathfrak{D}_{p}^{a}(\boldsymbol{\alpha}), \mathfrak{D}_{p}^{a}(\boldsymbol{\beta})}(t)\right)=0$. So that, $t \in S_{\boldsymbol{\alpha}_{a, r}, \boldsymbol{\beta}_{a, r}, p}$.

- Suppose that $t \bmod p \equiv 1-\mathfrak{D}_{p}^{a}\left(\beta_{s}\right) \bmod p$ with $s \in \mathcal{P}_{\mathfrak{D}_{p}^{a-1}(\boldsymbol{\beta}), r}$. Then $\beta_{s, a, r}=$ $\mathfrak{D}_{p}^{a}\left(\beta_{s}\right)+1$. We want to see that $t-1 \in S_{\boldsymbol{\alpha}_{a, r}, \boldsymbol{\beta}_{a, r}, p}$. To this end, we first prove that $t-1 \in E_{\boldsymbol{\alpha}_{a, r}, \boldsymbol{\beta}_{a, r}, p}$.

Assume for contradiction that $t=0$. Then, $1-\mathfrak{D}_{p}^{a}\left(\beta_{s}\right) \in p \mathbb{Z}_{(p)}$. Since, $\beta_{n}=1$ and, for all integers $m \geqslant 1, \mathfrak{D}_{p}^{m}(1)=1$, it follows that $\mathfrak{D}_{p}^{a}\left(\beta_{n}\right)-\mathfrak{D}_{p}^{a}\left(\beta_{s}\right) \in p \mathbb{Z}_{(p)}$. Then, from (P3), we obtain $\beta_{n}=\beta_{s}$. Thus, $1=\beta_{s}$. As $s \in \mathcal{P}_{\mathfrak{D}_{p}^{a-1}(\boldsymbol{\beta}), r}$ then, by definition of the set $\mathcal{P}_{\mathfrak{D}_{p}^{a-1}(\boldsymbol{\beta}), r}$, we have $\left(\mathfrak{D}_{p}^{a-1}\left(\beta_{s}\right)\right)_{r} \in p \mathbb{Z}_{(p)}$. Thus, $(1)_{r} \in p \mathbb{Z}_{(p)}$. But, $(1)_{r} \notin p \mathbb{Z}_{(p)}$ because $r \in\{0, \ldots, p-1\}$. Consequently, $t>0$. Thus, $t-1 \in\{0, \ldots, p-1\}$. As $t-1 \bmod p \equiv-\mathfrak{D}_{p}^{a}\left(\beta_{s}\right) \bmod p$ and $\beta_{s, a, r}=\mathfrak{D}_{p}^{a}\left(\beta_{s}\right)+1$ then $t-1 \in E_{\boldsymbol{\alpha}_{a, r}, \boldsymbol{\beta}_{a, r}, p}$.

Now, we have the following equality

$$
\begin{equation*}
\mathcal{Q}_{\mathfrak{D}_{p}^{a}(\boldsymbol{\alpha}), \mathfrak{D}_{p}^{a}(\boldsymbol{\beta})}(t)=\mathcal{Q}_{\boldsymbol{\alpha}_{a, r}, \boldsymbol{\beta}_{a, r}}(t-1) \cdot \frac{\prod_{w \in \mathcal{C}_{\mathfrak{O}_{p}^{a-1}(\boldsymbol{\alpha}), r}}\left(\mathfrak{D}_{p}^{a}\left(\alpha_{w}\right)+t-1\right)}{\prod_{w \in \mathcal{C}_{\mathfrak{D}_{p}^{a-1}(\boldsymbol{\beta}), r}}\left(\mathfrak{D}_{p}^{a}\left(\beta_{w}\right)+t-1\right)} \cdot \frac{\prod_{w \in \mathcal{P}_{\mathfrak{O}_{p}^{a-1}(\boldsymbol{\alpha}), r}} \mathfrak{D}_{p}^{a}\left(\alpha_{w}\right)}{\prod_{\mathcal{P}_{p}^{a-1}(\boldsymbol{\beta}), r}} \mathfrak{D}_{p}^{a}\left(\beta_{w}\right) . \tag{7.4}
\end{equation*}
$$

By $(\mathbf{P} 1)$, we know that $\mathfrak{D}_{p}^{a}\left(\alpha_{1}\right), \ldots, \mathfrak{D}_{p}^{a}\left(\alpha_{n}\right), \mathfrak{D}_{p}^{a}\left(\beta_{1}\right), \ldots, \mathfrak{D}_{p}^{a}\left(\beta_{n}\right)$ belong to $\mathbb{Z}_{(p)}^{*}$. Then,

$$
\left(\prod_{w \in \mathcal{P}_{\mathfrak{D}_{p}^{a-1}(\boldsymbol{\alpha}), r}} \mathfrak{D}_{p}^{a}\left(\alpha_{w}\right)\right) /\left(\prod_{w \in \mathcal{P}_{\mathfrak{D}_{p}^{a-1}(\boldsymbol{\beta}), r}} \mathfrak{D}_{p}^{a}\left(\beta_{w}\right)\right) \in \mathbb{Z}_{(p)}^{*}
$$

Assume for contradiction that there is $\gamma \in\left\{\alpha_{1}, \ldots, \alpha_{n}\right\}$ such that $\mathfrak{D}_{p}^{a}(\gamma)+t-1$ belongs to $p \mathbb{Z}_{(p)}$. Since $t-1 \bmod p \equiv-\mathfrak{D}_{p}^{a}\left(\beta_{s}\right) \bmod p$, it follows that $\mathfrak{D}_{p}^{a}(\gamma)-\mathfrak{D}_{p}^{a}\left(\beta_{s}\right) \in p \mathbb{Z}_{(p)}$. That is a contradiction because, according to (P2), $\mathfrak{D}_{p}^{a}(\gamma)-\mathfrak{D}_{p}^{a}\left(\beta_{s}\right) \notin p \mathbb{Z}_{(p)}$. For this reason, the elements $\mathfrak{D}_{p}^{a}\left(\alpha_{1}\right)+t-1, \ldots, \mathfrak{D}_{p}^{a}\left(\alpha_{n}\right)+t-1$ belong to $\mathbb{Z}_{(p)}^{*}$.

Again, suppose, to derive a contradiction, that there is $w \in \mathcal{C}_{\mathfrak{D}_{p}^{a-1}(\boldsymbol{\beta}), r}$ such that $\mathfrak{D}_{p}^{a}\left(\beta_{w}\right)+$ $t-1$ belongs to $p \mathbb{Z}_{(p)}$. Since $t-1 \equiv-\mathfrak{D}_{p}^{a}\left(\beta_{s}\right) \bmod p$, we obtain $\mathfrak{D}_{p}^{a}\left(\beta_{w}\right)-\mathfrak{D}_{p}^{a}\left(\beta_{s}\right) \in p \mathbb{Z}_{(p)}$. Then, according to $(\mathbf{P} 3)$, we have $\beta_{w}=\beta_{s}$. On the one hand, we have $\left(\mathfrak{D}_{p}^{a-1}\left(\beta_{s}\right)\right)_{r} \in p \mathbb{Z}_{(p)}$ because $s \in \mathcal{P}_{\mathfrak{D}_{p}^{a-1}(\boldsymbol{\beta}), r}$. On the other hand, $\left(\mathfrak{D}_{p}^{a-1}\left(\beta_{s}\right)\right)_{r} \notin p \mathbb{Z}_{(p)}$ because $\beta_{w}=\beta_{s}$ and $w \in \mathcal{C}_{\mathfrak{D}_{p}^{a-1}(\boldsymbol{\beta}), r}$. So that, we have a contradiction. For this reason, for every $w \in \mathcal{C}_{\mathfrak{D}_{p}^{a-1}(\boldsymbol{\beta}), r}$, $\mathfrak{D}_{p}^{a}\left(\beta_{w}\right)+t-1$ belongs to $\mathbb{Z}_{(p)}^{*}$. Consequently, the element

$$
\left(\prod_{w \in \mathcal{C}_{\mathfrak{D}_{p}^{a-1}(\boldsymbol{\alpha}), r}}\left(\mathfrak{D}_{p}^{a}\left(\alpha_{w}\right)+t-1\right)\right) /\left(\prod_{w \in \mathcal{C}_{\mathfrak{D}_{p}^{a-1}(\boldsymbol{\beta}), r}}\left(\mathfrak{D}_{p}^{a}\left(\beta_{w}\right)+t-1\right)\right) \in \mathbb{Z}_{(p)}^{*}
$$

Thus, from Equation (7.4), we have $v_{p}\left(\mathcal{Q}_{\boldsymbol{\alpha}_{a, r}, \boldsymbol{\beta}_{a, r}}(t-1)\right)=0$ because $v_{p}\left(\mathcal{Q}_{\mathfrak{D}_{p}^{a}(\boldsymbol{\alpha}), \mathfrak{D}_{p}^{a}(\boldsymbol{\beta})}(t)\right)=$ 0. So that, $t-1 \in S_{\boldsymbol{\alpha}_{a, r}, \boldsymbol{\beta}_{a, r}, p}$.

Consequently, $\tau$ is well-defined and it is clear that $\tau$ is the inverse of $\sigma$. Therefore, $\sigma$ is a bijective map.
B) Let $(a, r)$ be in $\{1, \ldots, l\} \times\{0, \ldots, p-1\}$, and let $t$ be in $S_{\boldsymbol{\alpha}_{a, r}, \boldsymbol{\beta}_{a, r}, p}$. We are going to see that $\mathcal{P}_{\boldsymbol{\alpha}_{a, r}, t}=\mathcal{P}_{\mathfrak{D}_{p}^{a}(\boldsymbol{\alpha}), \sigma(t)}$. As $t \in S_{\boldsymbol{\alpha}_{a, r}, \boldsymbol{\beta}_{a, r}, p}$ then $t \bmod p \equiv 1-\beta_{s, a, r}$ for some
$s \in\{1, \ldots, n\}$. In particular,

$$
t \bmod p \equiv\left\{\begin{array}{lll}
1-\mathfrak{D}_{p}^{a}\left(\beta_{s}\right) \bmod p & \text { if } & s \in \mathcal{C}_{\mathfrak{D}_{p}^{a-1}(\boldsymbol{\beta}), r} \\
-\mathfrak{D}_{p}^{a}\left(\beta_{s}\right) \bmod p & \text { if } & s \in \mathcal{P}_{\mathfrak{D}_{p}^{a-1}(\boldsymbol{\beta}), r}
\end{array}\right.
$$

- Suppose that $s \in \mathcal{C}_{\mathfrak{D}_{p}^{a-1}(\boldsymbol{\beta}), r}$. Then, $t \bmod p \equiv 1-\mathfrak{D}_{p}^{a}\left(\beta_{s}\right) \bmod p$ and $\sigma(t)=t$. We are going to see that $\mathcal{P}_{\boldsymbol{\alpha}_{a, r}, t} \subset \mathcal{P}_{\mathfrak{D}_{p}^{a}(\boldsymbol{\alpha}), t}$. Let $w$ be in $\mathcal{P}_{\boldsymbol{\alpha}_{a, r}, t}$. Then, $\left(\alpha_{w, a, r}\right)_{t} \in p \mathbb{Z}_{(p)}$. If $w \in \mathcal{C}_{\mathfrak{D}_{p}^{a-1}(\boldsymbol{\alpha}), r}$ then $\alpha_{w, a, r}=\mathfrak{D}_{p}^{a}\left(\alpha_{w}\right)$. So, $\left(\mathfrak{D}_{p}^{a}\left(\alpha_{w}\right)\right)_{t} \in p \mathbb{Z}_{(p)}$. Hence, $w \in \mathcal{P}_{\mathfrak{D}_{p}^{a}(\boldsymbol{\alpha}), t}$. Now, suppose that $w \in \mathcal{P}_{\mathfrak{D}_{p}^{a-1}(\boldsymbol{\alpha}), r}$. Then, $\alpha_{w, a, r}=1+\mathfrak{D}_{p}^{a}\left(\alpha_{w}\right)$. Thus, $\left(\mathfrak{D}_{p}^{a}\left(\alpha_{w}\right)+1\right)_{t} \in p \mathbb{Z}_{(p)}$. Suppose, towards a contradiction, that $\left(\mathfrak{D}_{p}^{a}\left(\alpha_{w}\right)\right)_{t} \notin p \mathbb{Z}_{(p)}$. Since $\left(\mathfrak{D}_{p}^{a}\left(\alpha_{w}\right)+1\right)_{t} \in p \mathbb{Z}_{(p)}$, we get $\mathfrak{D}_{p}^{a}\left(\alpha_{w}\right)+t \in p \mathbb{Z}_{(p)}$. As $0 \leqslant t \leqslant p-1$ then $t=p \mathfrak{D}_{p}^{a+1}\left(\alpha_{w}\right)-\mathfrak{D}_{p}^{a}\left(\alpha_{w}\right)$. As $t \bmod p \equiv$ $1-\mathfrak{D}_{p}^{a}\left(\beta_{s}\right) \bmod p$ then $1-\mathfrak{D}_{p}^{a}\left(\beta_{s}\right)+\mathfrak{D}_{p}^{a}\left(\alpha_{w}\right) \in p \mathbb{Z}_{(p)}$. This leads to a contradiction of $(\mathbf{P} 5)$. Whence, $\left(\mathfrak{D}_{p}^{a}\left(\alpha_{w}\right)\right)_{t} \in p \mathbb{Z}_{(p)}$. For this reason, $w \in \mathcal{P}_{\mathfrak{D}_{p}^{a}(\boldsymbol{\alpha}), t}$. Consequently, we have $\mathcal{P}_{\boldsymbol{\alpha}_{a, r}, t} \subset \mathcal{P}_{\mathfrak{D}_{p}^{a}(\boldsymbol{\alpha}), t}$.

Now, we show that $\mathcal{P}_{\mathfrak{D}_{p}^{a}(\boldsymbol{\alpha}), t} \subset \mathcal{P}_{\boldsymbol{\alpha}_{a, r}, t}$. Let $w$ be in $\mathcal{P}_{\mathfrak{D}_{p}^{a}(\boldsymbol{\alpha}), t}$. Then, $\left(\mathfrak{D}_{p}^{a}\left(\alpha_{w}\right)\right)_{t} \in p \mathbb{Z}_{(p)}$. If $w \in \mathcal{C}_{\mathfrak{D}_{p}^{a-1}(\boldsymbol{\alpha}), r}$ then $\alpha_{w, a, r}=\mathfrak{D}_{p}^{a}\left(\alpha_{w}\right)$. Thus, $w \in \mathcal{P}_{\boldsymbol{\alpha}_{a, r}, t}$. Now, if $w \in \mathcal{P}_{\mathfrak{D}_{p}^{a-1}(\boldsymbol{\alpha}), r}$ then $\alpha_{w, a, r}=1+\mathfrak{D}_{p}^{a}\left(\alpha_{w}\right)$. We have $\left(\mathfrak{D}_{p}^{a}\left(\alpha_{w}\right)\right)_{t} \in p \mathbb{Z}_{(p)}$ and from ( $\mathbf{P} 1$ ) we know that $\mathfrak{D}_{p}^{a}\left(\alpha_{w}\right) \in$ $\mathbb{Z}_{(p)}^{*}$. Thus, $\left(\mathfrak{D}_{p}^{a}\left(\alpha_{w}\right)+1\right)_{t} \in p \mathbb{Z}_{(p)}$. Then, $w \in \mathcal{P}_{\boldsymbol{\alpha}_{a, r}, t}$. Consequently, we have $\mathcal{P}_{\mathfrak{D}_{p}^{a}(\boldsymbol{\alpha}), t} \subset$ $\mathcal{P}_{\boldsymbol{\alpha}_{a, r}, t}$.

Therefore, $\mathcal{P}_{\boldsymbol{\alpha}_{a, r}, t}=\mathcal{P}_{\mathfrak{D}_{p}^{a}(\boldsymbol{\alpha}), t}$. But, remember that $\sigma(t)=t$. Whence, we obtain $\mathcal{P}_{\boldsymbol{\alpha}_{a, r}, t}=$ $\mathcal{P}_{\mathfrak{D}_{p}^{a}(\boldsymbol{\alpha}), \sigma(t)}$.

- Suppose that $s \in \mathcal{P}_{\mathfrak{D}_{p}^{a-1}(\boldsymbol{\beta}), r}$. Then $t \bmod p \equiv-\mathfrak{D}_{p}^{a}\left(\beta_{s}\right) \bmod p$ and $\sigma(t)=t+1$. We are going to see that $\mathcal{P}_{\boldsymbol{\alpha}_{a, r}, t} \subset \mathcal{P}_{\mathfrak{D}_{p}^{a}(\boldsymbol{\alpha}), t+1}$. Let $w$ be in $\mathcal{P}_{\boldsymbol{\alpha}_{a, r}, t}$. Then $\left(\alpha_{w, a, r}\right)_{t} \in p \mathbb{Z}_{(p)}$. Suppose that $w \in \mathcal{C}_{\mathfrak{D}_{p}^{a-1}(\boldsymbol{\alpha}), r}$. Then, $\alpha_{w, a, r}=\mathfrak{D}_{p}^{a}\left(\alpha_{w}\right)$. So that, $\left(\mathfrak{D}_{p}^{a}\left(\alpha_{w}\right)\right)_{t} \in p \mathbb{Z}_{(p)}$. For this reason, $\left(\mathfrak{D}_{p}^{a}\left(\alpha_{w}\right)\right)_{t+1} \in p \mathbb{Z}_{(p)}$. Therefore, $w \in \mathcal{P}_{\mathfrak{D}_{p}^{a}(\boldsymbol{\alpha}), t+1}$. Suppose now that $w \in \mathcal{P}_{\mathfrak{D}_{p}^{a-1}(\boldsymbol{\alpha}), r}$. Then, $\alpha_{w, a, r}=1+\mathfrak{D}_{p}^{a}\left(\alpha_{w}\right)$. So that, $\left(\mathfrak{D}_{p}^{a}\left(\alpha_{w}\right)+1\right)_{t} \in p \mathbb{Z}_{(p)}$. Thus, $\left(\mathfrak{D}_{p}^{a}\left(\alpha_{w}\right)\right)_{t+1} \in p \mathbb{Z}_{(p)}$. Hence, $w \in \mathcal{P}_{\mathfrak{D}_{p}^{a}(\boldsymbol{\alpha}), t+1}$. Consequently, $\mathcal{P}_{\boldsymbol{\alpha}_{a, r}, t} \subset \mathcal{P}_{\mathfrak{D}_{p}^{a}(\boldsymbol{\alpha}), t+1}$.

Now, we show that $\mathcal{P}_{\mathfrak{D}_{p}^{a}(\boldsymbol{\alpha}), t+1} \subset \mathcal{P}_{\boldsymbol{\alpha}_{a, r}, t}$. Let $w$ be in $\mathcal{P}_{\mathfrak{D}_{p}^{a}(\boldsymbol{\alpha}), t+1}$. Then, $\left(\mathfrak{D}_{p}^{a}\left(\alpha_{w}\right)\right)_{t+1} \in$ $p \mathbb{Z}_{(p)}$. Suppose that $w \in \mathcal{C}_{\mathfrak{D}_{p}^{a-1}(\boldsymbol{\alpha}), r}$. Then, $\alpha_{w, a, r}=\mathfrak{D}_{p}^{a}\left(\alpha_{w}\right)$. Assume for contradiction that $\left(\mathfrak{D}_{p}^{a}\left(\alpha_{w}\right)\right)_{t} \notin p \mathbb{Z}_{(p)}$. Thus, $\mathfrak{D}_{p}^{a}\left(\alpha_{w}\right)+t \in p \mathbb{Z}_{(p)}$. As $0 \leqslant t \leqslant p-1$ then $t=p \mathfrak{D}_{p}^{a+1}\left(\alpha_{w}\right)-\mathfrak{D}_{p}^{a}\left(\alpha_{w}\right)$. Since $t \bmod p \equiv-\mathfrak{D}_{p}^{a}\left(\beta_{s}\right) \bmod p$, it follows that $\mathfrak{D}_{p}^{a}\left(\alpha_{w}\right)-\mathfrak{D}_{p}^{a}\left(\beta_{s}\right) \in p \mathbb{Z}_{(p)}$. This contradicts our condition ( $\mathbf{P} 2$ ). Then, we have $\left(\mathfrak{D}_{p}^{a}\left(\alpha_{w}\right)\right)_{t} \in p \mathbb{Z}_{(p)}$. For this reason, $w \in \mathcal{P}_{\boldsymbol{\alpha}_{a, r}, t}$. Now, suppose that $w \in \mathcal{P}_{\mathfrak{D}_{p}^{a-1}(\boldsymbol{\alpha}), r}$. Then, $\alpha_{w, a, r}=1+\mathfrak{D}_{p}^{a}\left(\alpha_{w}\right)$. By $(\mathbf{P} 1)$, we know that $\mathfrak{D}_{p}^{a}\left(\alpha_{w}\right) \in$ $\mathbb{Z}_{(p)}^{*}$ and since $\left(\mathfrak{D}_{p}^{a}\left(\alpha_{w}\right)\right)_{t+1} \in p \mathbb{Z}_{(p)}$, it follows that $\left(\mathfrak{D}_{p}^{a}\left(\alpha_{w}\right)+1\right)_{t} \in p \mathbb{Z}_{(p)}$. So that $w \in$ $\mathcal{P}_{\boldsymbol{\alpha}_{a, r}, t}$. Consequently, $\mathcal{P}_{\mathfrak{D}_{p}^{a}(\boldsymbol{\alpha}), t+1} \subset \mathcal{P}_{\boldsymbol{\alpha}_{a, r}, t}$.

Therefore, $\mathcal{P}_{\boldsymbol{\alpha}_{a, r}, t}=\mathcal{P}_{\mathfrak{D}_{p}^{a}(\boldsymbol{\alpha}), t+1}$. But, remember that $\sigma(t)=t+1$. Whence, we obtain $\mathcal{P}_{\boldsymbol{\alpha}_{a, r}, t}=\mathcal{P}_{\mathfrak{D}_{p}^{a}(\boldsymbol{\alpha}), \sigma(t)}$.

Thus, for every $t \in S_{\boldsymbol{\alpha}_{a, r}, \boldsymbol{\beta}_{a, r}, p}$, we have $\mathcal{P}_{\boldsymbol{\alpha}_{a, r}, t}=\mathcal{P}_{\mathfrak{D}_{p}^{a}(\boldsymbol{\alpha}), \sigma(t)}$. One shows, in an exactly similar way that, for every $t \in S_{\boldsymbol{\alpha}_{a, r}, \boldsymbol{\beta}_{a, r}, p}$, we have $\mathcal{P}_{\boldsymbol{\beta}_{a, r}, t}=\mathcal{P}_{\mathfrak{D}_{p}^{a}(\boldsymbol{\beta}), \sigma(t)}$.

Finally, for every $t \in S_{\boldsymbol{\alpha}_{a, r}, \boldsymbol{\beta}_{a, r}, p}$, we also have $\mathcal{C}_{\boldsymbol{\alpha}_{a, r}, t}=\mathcal{C}_{\mathfrak{D}_{p}^{a}(\boldsymbol{\alpha}), \sigma(t)}$ because $\mathcal{C}_{\boldsymbol{\alpha}_{a, r}, t}$ is the complement of $\mathcal{P}_{\boldsymbol{\alpha}_{a, r}, t}$ in $\{1, \ldots, n\}, \mathcal{C}_{\mathfrak{D}_{p}^{a}(\boldsymbol{\alpha}), \sigma(t)}$ is the complement of $\mathcal{P}_{\mathfrak{D}_{p}^{a}(\boldsymbol{\alpha}), \sigma(t)}$ in $\{1, \ldots, n\}$, and we have already seen that $\mathcal{P}_{\boldsymbol{\alpha}_{a, r}, t}=\mathcal{P}_{\mathfrak{D}_{p}^{a}(\boldsymbol{\alpha}), \sigma(t)}$. In a similar way one has $\mathcal{C}_{\boldsymbol{\beta}_{a, r}, t}=\mathcal{C}_{\mathfrak{D}_{p}^{a}(\boldsymbol{\beta}), \sigma(t)}$.

## 8. Proof of Lemma 6.2

Lemma 6.2 is obtained from the following lemma
Lemma 8.1.-Let $\boldsymbol{\alpha}=\left(\alpha_{1}, \ldots, \alpha_{n}\right), \boldsymbol{\beta}=\left(\beta_{1}, \ldots, \beta_{n-1}, 1\right)$ be in $\left(\mathbb{Q} \backslash \mathbb{Z}_{\leqslant 0}\right)^{n}$ and let $p$ be a prime number such that $p>d_{\boldsymbol{\alpha}, \boldsymbol{\beta}}$ and $f(z):={ }_{n} F_{n-1}(\boldsymbol{\alpha}, \boldsymbol{\beta} ; z)$ belongs to $\mathbb{Z}_{(p)}[[z]]$. Suppose that $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ satisfies the $\boldsymbol{P}_{p, l}$ property, where $l$ is the order of $p$ in $\left(\mathbb{Z} / d_{\boldsymbol{\alpha}, \boldsymbol{\beta}} \mathbb{Z}\right)^{*}$. Then, for each $r \in S_{\boldsymbol{\alpha}, \boldsymbol{\beta}, p}, f_{1, r} \in 1+z \mathbb{Z}_{(p)}[[z]]$ and

$$
f(z) \equiv \sum_{r \in S_{\boldsymbol{\alpha}, \boldsymbol{\beta}, p}} Q_{r} f_{1, r}^{p} \bmod p \quad \text { with } \quad Q_{r}(z)=\sum_{s=r}^{r^{\prime}-1} \mathcal{Q}_{\boldsymbol{\alpha}, \boldsymbol{\beta}}(s) z^{s}
$$

where $r^{\prime}$ is defined as follows. If $r \neq \max E_{\boldsymbol{\alpha}, \boldsymbol{\beta}, p}$ then $r^{\prime}$ is the element in $E_{\boldsymbol{\alpha}, \boldsymbol{\beta}, p}$ such that $r<r^{\prime}$ and $\left(r, r^{\prime}\right) \cap E_{\boldsymbol{\alpha}, \boldsymbol{\beta}, p}=\emptyset$ or otherwise, $r^{\prime}=p$.

Proof of Lemma 6.2. - We proceed by induction on $a \in\{1, \ldots, l\}$. Suppose $a=1$. By assumption, $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ satisfies the $\mathbf{P}_{p, l}$ property and $f(z) \in \mathbb{Z}_{(p)}[[z]]$. Thus, the hypotheses of Lemma 8.1 are satisfied and we conclude that

$$
f(z) \equiv \sum_{r \in S_{\boldsymbol{\alpha}, \boldsymbol{\beta}, p}} Q_{r} f_{r}^{p} \bmod p
$$

where, each $Q_{r}(z)$ belongs to $\mathbb{Z}_{(p)}$ and has degree less than $p$, and $f_{r}(z) \in 1+z \mathbb{Z}_{(p)}[[z]]$. We now suppose that the conclusion of our lemma is true for some $a$ in $\{1, \ldots, l-1\}$. We are going to prove that it is also true for $a+1$. By induction hypothesis, we have

$$
\begin{equation*}
f \equiv \sum_{r \in S_{\mathfrak{O}_{p}^{a-1}(\alpha), \mathcal{D}_{p}^{a-1}(\boldsymbol{\beta}), p}} Q_{a, r}(z) f_{a, r}^{p^{a}} \bmod p \tag{8.1}
\end{equation*}
$$

where, for every $r \in S_{\mathfrak{D}_{p}^{a-1}(\boldsymbol{\alpha}), \mathfrak{D}_{p}^{a-1}(\boldsymbol{\beta}), p}, Q_{a, r}(z)$ belongs to $\mathbb{Z}_{(p)}[z]$ and has degree less than $p^{a}$ and $f_{a, r} \in 1+z \mathbb{Z}_{(p)}[[z]]$.

We fix $r$ in $S_{\mathfrak{D}_{p}^{a-1}(\boldsymbol{\alpha}), \mathfrak{D}_{p}^{a-1}(\boldsymbol{\beta}), p}$. By definition, $f_{a, r}(z)={ }_{n} F_{n-1}\left(\boldsymbol{\alpha}_{a, r}, \boldsymbol{\beta}_{a, r} ; z\right)$. We would like to apply Lemma 8.1 to $f_{a, r}(z)$. To this end, we are going to see that the hypotheses of Lemma 8.1 are satisfied. By induction hypothesis, we know that $f_{a, r}(z) \in \mathbb{Z}_{(p)}[[z]]$ and thanks to ( $\mathbf{P} 1$ ), $\boldsymbol{\alpha}_{a, r}, \boldsymbol{\beta}_{a, r}$ belong to ( $\left.\mathbb{Z}_{(p)} \backslash \mathbb{Z}_{\leqslant 0}\right)^{n}$. According to Remark 6.4, ( $\boldsymbol{\alpha}_{a, r}, \boldsymbol{\beta}_{a, r}$ ) satisfies the $\mathbf{P}_{p, l^{\prime}}$ property, where $l^{\prime}$ is the order of $p$ in $\left(\mathbb{Z} / d_{\boldsymbol{\alpha}_{a, r}, \boldsymbol{\beta}_{a, r}} \mathbb{Z}\right)^{*}$. We can then apply Lemma 8.1 to $f_{a, r}$ and we obtain

$$
\begin{equation*}
f_{a, r}(z) \equiv \sum_{\mu \in S_{\boldsymbol{\alpha}_{a, r}, \boldsymbol{\beta}_{a, r}, p}} P_{\mu}(z) g_{\mu}^{p} \bmod p \tag{8.2}
\end{equation*}
$$

where, each $P_{\mu}(z) \in \mathbb{Z}_{(p)}[z]$ has degree less than $p$, and

$$
g_{\mu}(z)=\sum_{m \geqslant 0}\left(\frac{\prod_{s \in \mathcal{C}_{\alpha_{a, r}, \mu}} \mathfrak{D}_{p}\left(\alpha_{s, a, r}\right)_{m} \prod_{s \in \mathcal{P}_{\alpha_{a, r}, \mu}}\left(\mathfrak{D}_{p}\left(\alpha_{s, a, r}\right)+1\right)_{m}}{\prod_{s \in \mathcal{C}_{\boldsymbol{\beta}_{a, r}, \mu}} \mathfrak{D}_{p}\left(\beta_{s, a, r}\right)_{m} \prod_{s \in \mathcal{P}_{\boldsymbol{\beta}_{a, r}, \mu}}\left(\mathfrak{D}_{p}\left(\beta_{s, a, r}\right)+1\right)_{m}}\right) z^{m} \in 1+z \mathbb{Z}_{(p)}[[z]] .
$$

By (2) of Remark 6.3, we know that $\mathfrak{D}_{p}\left(\boldsymbol{\alpha}_{a, r}\right)=\mathfrak{D}_{p}^{a+1}(\boldsymbol{\alpha})$ and that $\mathfrak{D}_{p}\left(\boldsymbol{\beta}_{a, r}\right)=\mathfrak{D}_{p}^{a+1}(\boldsymbol{\beta})$. Hence, it follows that, for all $\mu \in S_{\boldsymbol{\alpha}_{a, r}, \boldsymbol{\beta}_{a, r}, p}$,

$$
g_{\mu}(z)=\sum_{m \geqslant 0}\left(\frac{\prod_{s \in \mathcal{C}_{\alpha_{a, r}, \mu}} \mathfrak{D}_{p}^{a+1}\left(\alpha_{s}\right)_{m} \prod_{s \in \mathcal{P}_{\alpha_{a, r}, \mu}}\left(\mathfrak{D}_{p}^{a+1}\left(\alpha_{s}\right)+1\right)_{m}}{\prod_{s \in \mathcal{C}_{\boldsymbol{\beta}_{a, r}, \mu}} \mathfrak{D}_{p}^{a+1}\left(\beta_{s}\right)_{m} \prod_{s \in \mathcal{P}_{\boldsymbol{\beta}_{a, r}, \mu}}\left(\mathfrak{D}_{p}^{a+1}\left(\beta_{s}\right)+1\right)_{m}}\right) z^{m} .
$$

We now want to see that $g_{\mu}(z)=f_{a+1, \sigma(\mu)}$ for all $\mu \in S_{\boldsymbol{\alpha}_{a, r}, \boldsymbol{\beta}_{a, r}, p}$, where $\sigma: S_{\boldsymbol{\alpha}_{a, r}, \boldsymbol{\beta}_{a, r}, p} \rightarrow$ $S_{\mathfrak{D}_{p}^{a}(\boldsymbol{\alpha}), \mathfrak{D}_{p}^{a}(\boldsymbol{\beta}), p}$ is the bijective map given by Lemma 6.1. By definition, we have $f_{a+1, \gamma}=$ ${ }_{n} F_{n-1}\left(\boldsymbol{\alpha}_{a+1, \gamma}, \boldsymbol{\beta}_{a+1, \gamma} ; z\right)$ for all $\gamma \in S_{\mathfrak{D}_{p}^{a}(\boldsymbol{\alpha}), \mathfrak{D}_{p}^{a}(\boldsymbol{\beta}), p}$. Thus,

$$
f_{a+1, \gamma}=\sum_{m \geqslant 0}\left(\frac{\prod_{s \in \mathcal{C}_{\mathfrak{D}_{p}^{a}(\alpha), \gamma}} \mathfrak{D}_{p}^{a+1}\left(\alpha_{s}\right)_{m} \prod_{s \in \mathcal{P}_{\mathfrak{D}_{p}^{a}(\alpha), \gamma}}\left(\mathfrak{D}_{p}^{a+1}\left(\alpha_{s}\right)+1\right)_{m}}{\prod_{\mathcal{D}_{p}^{a}(\boldsymbol{\beta}), \gamma}} \mathfrak{D}_{p}^{a+1}\left(\beta_{s}\right)_{m} \prod_{s \in \mathcal{P}_{\mathfrak{D}_{p}^{a}(\boldsymbol{\beta}), \gamma}}\left(\mathfrak{D}_{p}^{a+1}\left(\beta_{s}\right)+1\right)_{m}\right) z^{m}
$$

By invoking B) of Lemma 6.1, we obtain, for every $\mu \in S_{\boldsymbol{\alpha}_{a, r}, \boldsymbol{\beta}_{a, r}, p}$, the following equalities $\mathcal{P}_{\boldsymbol{\alpha}_{a, r}, \mu}=\mathcal{P}_{\mathfrak{D}_{p}^{a}(\boldsymbol{\alpha}), \sigma(\mu)}, \mathcal{C}_{\boldsymbol{\alpha}_{a, r}, \mu}=\mathcal{C}_{\mathfrak{D}_{p}^{a}(\boldsymbol{\alpha}), \sigma(\mu)}, \mathcal{P}_{\boldsymbol{\beta}_{a, r}, \mu}=\mathcal{P}_{\mathfrak{D}_{p}^{a}(\boldsymbol{\beta}), \sigma(\mu)}$, and $\mathcal{C}_{\boldsymbol{\beta}_{a, r}, \mu}=$ $\mathcal{C}_{\mathfrak{D}_{p}^{a}(\boldsymbol{\beta}), \sigma(\mu)}$. For this reason, $g_{\mu}(z)=f_{a+1, \sigma(\mu)}$. By A) of Lemma 6.1, we know that $\sigma$ : $S_{\boldsymbol{\alpha}_{a, r}, \boldsymbol{\beta}_{a, r}, p} \rightarrow S_{\mathfrak{D}_{p}^{a}(\boldsymbol{\alpha}), \mathfrak{D}_{p}^{a}(\boldsymbol{\beta}), p}$ is bijective. Then, from (8.2), we infer that

$$
\begin{equation*}
f_{a, r}(z) \equiv \sum_{\gamma \in S_{\mathfrak{D}_{p}^{a}(\alpha), \mathbb{D}_{p}^{a}(\boldsymbol{\beta}), p}} T_{r, \gamma}(z) f_{a+1, \gamma}^{p} \bmod p \tag{8.3}
\end{equation*}
$$

where $T_{r, \gamma}(z) \in \mathbb{Z}_{(p)}[z]$ with degree less than $p$. As $r$ in $S_{\mathfrak{D}_{p}^{a-1}(\boldsymbol{\alpha}), \mathfrak{D}_{p}^{a-1}(\boldsymbol{\beta}), p}$ is an arbitrary element, then, from Equalities (8.1) and (8.3), we obtain

$$
f(z) \equiv \sum_{\gamma \in S_{\mathfrak{O}_{p}^{a}(\boldsymbol{\alpha}), \mathfrak{D}_{p}^{a}(\boldsymbol{\beta}), p}} Q_{a+1, \gamma} f_{a+1, \gamma}^{p^{a+1}} \bmod p, \text { with } Q_{a+1, \gamma}=\sum_{r \in S_{\mathfrak{O}_{p}^{a-1}(\boldsymbol{\alpha}), \mathfrak{D}_{p}^{a-1}(\boldsymbol{\beta}), p}} Q_{a, r}(z) T_{r, \gamma}^{p^{a}}
$$

For every $\gamma \in S_{\mathfrak{D}_{p}^{a}(\boldsymbol{\alpha}), \mathfrak{D}_{p}^{a}(\boldsymbol{\beta}), p}$, the polynomial $Q_{a+1, \gamma}$ has degree less than $p^{a+1}$ because, for every $r \in S_{\mathfrak{O}_{p}^{a-1}(\boldsymbol{\alpha}), \mathfrak{D}_{p}^{a-1}(\boldsymbol{\beta}), p}, Q_{a, r} \in \mathbb{Z}_{(p)}[z]$ has degree less than $p^{a}$ and $T_{r, \gamma} \in \mathbb{Z}_{(p)}[z]$ has degree less than $p$. This completes the proof.

The remainder of this section is devoted to proving Lemma 8.1.

### 8.1. Cartier operators

The proof of the Lemma 8.1 depends essentially on Lemma 8.2. In preparation for stating Lemma 8.2, we recall the definition of Cartier operators over the ring $\mathbb{F}_{p}[[z]]$. For each
$r \in\{0, \ldots, p-1\}$, we have the $\mathbb{F}_{p}$-linear operator $\Lambda_{r}: \mathbb{F}_{p}[[z]] \rightarrow \mathbb{F}_{p}[[z]]$ given by

$$
\Lambda_{r}\left(\sum_{j \geqslant 0} a(n) z^{j}\right)=\sum_{j \geqslant 0} a(j p+r) z^{j}
$$

The operators $\Lambda_{0}, \ldots, \Lambda_{p-1}$ are called the Cartiers Operators ${ }^{(3)}$.
Lemma 8.2. - Let the assumptions be as in Lemma 8.1. Then

$$
f(z) \equiv \sum_{r \in S_{\boldsymbol{\alpha}, \boldsymbol{\beta}, p}} P_{r}(z) \Lambda_{r}(f)^{p} \bmod p \quad \text { with } \quad P_{r}(z)=\sum_{s=r}^{r^{\prime}-1} \frac{\mathcal{Q}_{\boldsymbol{\alpha}, \boldsymbol{\beta}}(s)}{\mathcal{Q}_{\boldsymbol{\alpha}, \boldsymbol{\beta}}(r)} z^{s}
$$

where $r^{\prime}$ is defined as follows. If $r \neq \max E_{\boldsymbol{\alpha}, \boldsymbol{\beta}, p}$ then $r^{\prime}$ is the element in $E_{\boldsymbol{\alpha}, \boldsymbol{\beta}, p}$ such that $r<r^{\prime}$ and $\left(r, r^{\prime}\right) \cap E_{\boldsymbol{\alpha}, \boldsymbol{\beta}, p}=\emptyset$ or otherwise, $r^{\prime}=p$.

### 8.2. Auxiliary result I

In order to prove Lemma 8.2, we need the next auxiliary results which we state and prove. These auxiliary results deal with $p$-adic properties of the sequence $\left\{\mathcal{Q}_{\boldsymbol{\alpha}, \boldsymbol{\beta}}(i)\right\}_{i \geqslant 0}$. The main result is Lemma 8.5.

Lemma 8.3. - Letp be a prime number and let $\boldsymbol{\alpha}=\left(\alpha_{1}, \ldots, \alpha_{n}\right), \boldsymbol{\beta}=\left(\beta_{1}, \ldots, \beta_{n-1}, 1\right)$ be in $\left(\mathbb{Z}_{(p)} \backslash \mathbb{Z}_{\leqslant 0}\right)^{n}$. Then, for all integers $j \geqslant 0$,

$$
\mathcal{Q}_{\boldsymbol{\alpha}, \boldsymbol{\beta}}(j p)=\mathcal{Q}_{\mathfrak{D}_{p}(\boldsymbol{\alpha}), \mathfrak{D}_{p}(\boldsymbol{\beta})}(j) \omega
$$

where $\omega \in \mathbb{Z}_{(p)}^{*}$ and $\omega \equiv 1 \bmod p$.
Proof. - If $j=0$ then there is nothing to prove. So, we suppose that $j>0$. Let $\gamma$ be in $\left\{\alpha_{1}, \ldots, \alpha_{n}, \beta_{1}, \ldots, \beta_{n}\right\}$ and let $r$ be the unique integer in $\{0,1, \ldots, p-1\}$ such that $p \mathfrak{D}_{p}(\gamma)-\gamma=r$. It is clear that

$$
(\gamma)_{j p}=\prod_{t=0}^{p-1}(\gamma+t) \prod_{t=0}^{p-1}(\gamma+p+t) \cdots \prod_{t=0}^{p-1}(\gamma+(j-1) p+t)
$$

Note that, for all nonnegative integers $s, \mathfrak{D}_{p}(\gamma+s p)=\mathfrak{D}_{p}(\gamma)+s$ because $p\left(\mathfrak{D}_{p}(\gamma)+s\right)-$ $(\gamma+s p)=r$. Then

$$
\begin{aligned}
(\gamma)_{j p}=\prod_{s=0}^{j-1}\left(p\left(\mathfrak{D}_{p}(\gamma)+s\right) \prod_{\substack{t=0 \\
t \neq r}}^{p-1}(\gamma+s p+t)\right) & =p^{j} \prod_{s=0}^{j-1}\left(\mathfrak{D}_{p}(\gamma)+s\right) \prod_{s=0}^{j-1}\left(\prod_{\substack{t=0 \\
t \neq r}}^{p-1}(\gamma+s p+t)\right) \\
& =p^{j}\left(\mathfrak{D}_{p}(\gamma)\right)_{j} \prod_{s=0}^{j-1}\left(\prod_{\substack{t=0 \\
t \neq r}}^{p-1}(\gamma+s p+t)\right)
\end{aligned}
$$

[^3]By Wilson's Theorem, it follows that, for all nonnegative integers $s$,

$$
\prod_{\substack{t=0 \\ t \neq r}}^{p-1}(\gamma+s p+t) \equiv(p-1)!\equiv-1 \bmod p
$$

Therefore,

$$
\begin{equation*}
(\gamma)_{j p}=p^{j}\left(\mathfrak{D}_{p}(\gamma)\right)_{j} \lambda \tag{8.4}
\end{equation*}
$$

where $\lambda \in \mathbb{Z}_{(p)}^{*}$ and $\lambda \equiv(-1)^{j} \bmod p$.
Since $\gamma$ is an arbitrary element in $\left\{\alpha_{1}, \ldots, \alpha_{n}, \beta_{1}, \ldots, \beta_{n}\right\}$ and

$$
\mathcal{Q}_{\boldsymbol{\alpha}, \boldsymbol{\beta}}(j p)=\frac{\left(\alpha_{1}\right)_{j p} \cdots\left(\alpha_{n}\right)_{j p}}{\left(\beta_{1}\right)_{j p} \cdots\left(\beta_{n-1}\right)_{j p}(1)_{j p}}
$$

it follows from Equation (8.4) that

$$
\begin{equation*}
\mathcal{Q}_{\boldsymbol{\alpha}, \boldsymbol{\beta}}(j p)=\mathcal{Q}_{\mathfrak{D}_{p}(\boldsymbol{\alpha}), \mathfrak{D}_{p}(\boldsymbol{\beta})}(j) \omega \tag{8.5}
\end{equation*}
$$

where $\omega \in \mathbb{Z}_{(p)}^{*}$ and $\omega \equiv 1 \bmod p$.

Lemma 8.4. - Let $p$ be a prime number, let $\alpha$ be in $\mathbb{Z}_{(p)}$ and let $n \geqslant 1$ be an integer. If $n=r_{0}+r_{1} p+\cdots+r_{s} p^{s}$ is the $p$-adic expansion of $n$ then

$$
(\alpha)_{n} \in p^{n_{1}}\left(\mathfrak{D}_{p}(\alpha)\right)_{n_{1}}\left(\alpha+n_{1} p\right)_{r_{0}} \mathbb{Z}_{(p)}^{*}
$$

where $n_{1}=r_{1}+r_{2} p+\cdots+r_{s} p^{s-1}$.
Proof. - It is not hard to see that $(\alpha)_{n}=\prod_{k=0}^{n_{1}-1}(\alpha+k p)_{p} \cdot\left(\alpha+n_{1} p\right)_{r_{0}}$. We know that there is a unique $r \in\{0, \ldots, p-1\}$ such that $\alpha+r=p \mathfrak{D}_{p}(\alpha)$. Then, for all integers $m \geqslant 1$, $\alpha+m p+r=p\left(\mathfrak{D}_{p}(\alpha)+m\right)$. Hence, for all $m \geqslant 1$,

$$
(\alpha+m p)_{p}=p\left(\mathfrak{D}_{p}(\alpha)+m\right) \prod_{\substack{i=0 \\ i \neq r}}^{p-1}(\alpha+m p+i)
$$

But, for all $0 \leqslant i<p$ such that $i \neq r,(\alpha+m p+i) \in \mathbb{Z}_{(p)}^{*}$ because $r$ is the unique element in $\{0, \ldots, p-1\}$ such that $\alpha+r \in p \mathbb{Z}_{(p)}$. So, we conclude that

$$
(\alpha)_{n} \in p^{n_{1}}\left(\mathfrak{D}_{p}(\alpha)\right)_{n_{1}}\left(\alpha+n_{1} p\right)_{r_{0}} \mathbb{Z}_{(p)}^{*}
$$

Lemma 8.5. - Let the assumptions be as in Lemma 8.1. If $v_{p}\left(\mathcal{Q}_{\boldsymbol{\alpha}, \boldsymbol{\beta}}(r)\right)>0$ then, for every $j \in \mathbb{N}, v_{p}\left(\mathcal{Q}_{\boldsymbol{\alpha}, \boldsymbol{\beta}}(j p+r)\right)>0$.

Proof. - We split the proof into five steps.
Step I: We will prove that, for all integers $a \geqslant 1, \mathfrak{D}_{p}^{a}\left(\alpha_{i}\right)-\mathfrak{D}_{p}^{a}\left(\beta_{j}\right) \in \mathbb{Z}_{(p)}^{*}$. Since $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ satisfies the $\mathbf{P}_{p, l}$ propery, we know that, for all $1 \leqslant m \leqslant l$, $\mathfrak{D}_{p}^{m}\left(\alpha_{i}\right)-\mathfrak{D}_{p}^{m}\left(\beta_{j}\right) \in \mathbb{Z}_{(p)}^{*}$ for all $1 \leqslant i, j \leqslant n$. Therefore, it is sufficient to prove that, for all $a \geqslant 1, \mathfrak{D}_{p}^{a}(\boldsymbol{\alpha})=\mathfrak{D}_{p}^{q}(\boldsymbol{\alpha})$ and $\mathfrak{D}_{p}^{a}(\boldsymbol{\beta})=\mathfrak{D}_{p}^{q}(\boldsymbol{\beta})$ where $q=a \bmod l$ with $1 \leqslant q<l$ if $a \neq 0 \bmod l$, and $q=l$ if
$a=0 \bmod l$. From the definition of $\mathfrak{D}_{p}$, it is not hard to see that, for all $a \geqslant 1, d_{\mathfrak{D}_{p}^{a}(\boldsymbol{\alpha}), \mathfrak{D}_{p}^{a}(\boldsymbol{\beta})}$ divides $d_{\boldsymbol{\alpha}, \boldsymbol{\beta}}$. Consequently, $p^{l} \equiv 1 \bmod d_{\mathfrak{D}_{p}^{a}(\boldsymbol{\alpha}), \mathfrak{D}_{p}^{a}(\boldsymbol{\beta})}$ for all $a \geqslant 1$. Further, for all $1 \leqslant$ $m \leqslant l, \mathfrak{D}_{p}^{m}(\boldsymbol{\alpha}), \mathfrak{D}_{p}^{m}(\boldsymbol{\beta}) \in\left(\mathbb{Z}_{(p)}^{*} \cap(0,1]\right)^{n}$ because, by assumption, $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ satisfies ( $\left.\mathbf{P} 1\right)$. So, by Lemma 2.1, we conclude that, for all $t \geqslant 1$ and $1 \leqslant m \leqslant l$, $\mathfrak{D}_{p}^{l t}\left(\mathfrak{D}_{p}^{m}(\boldsymbol{\alpha})\right)=\mathfrak{D}_{p}^{m}(\boldsymbol{\alpha})$ and $\mathfrak{D}_{p}^{l t}\left(\mathfrak{D}_{p}^{m}(\boldsymbol{\beta})\right)=\mathfrak{D}_{p}^{m}(\boldsymbol{\beta})$. Consequently, if $a=q+l t$ with $0 \leqslant q<l$ and $q \neq 0$ then $\mathfrak{D}_{p}^{a}(\boldsymbol{\alpha})=\mathfrak{D}_{p}^{q}(\boldsymbol{\alpha}), \mathfrak{D}_{p}^{a}(\boldsymbol{\beta})=\mathfrak{D}_{p}^{q}(\boldsymbol{\beta})$ and if $q=0, \mathfrak{D}_{p}^{a}(\boldsymbol{\alpha})=\mathfrak{D}_{p}^{(t-1) l}\left(\left(\mathfrak{D}_{p}^{l}(\boldsymbol{\alpha})\right)=\mathfrak{D}_{p}^{l}(\boldsymbol{\alpha})\right.$, $\mathfrak{D}_{p}^{a}(\boldsymbol{\beta})=\mathfrak{D}_{p}^{(t-1) l}\left(\left(\mathfrak{D}_{p}^{l}(\boldsymbol{\beta})\right)=\mathfrak{D}_{p}^{l}(\boldsymbol{\beta})\right.$.

Step II Let $(a, b)$ be in $\mathbb{Z}_{\geqslant 0} \times\{0, \ldots, p\}$ and let $\gamma \in\left\{\alpha_{1}, \ldots, \alpha_{n}, \beta_{1}, \ldots, \beta_{n-1}\right\}$. If $v_{p}\left(\mathfrak{D}_{p}^{a}(\gamma)_{b}\right) \geqslant 1$ then $v_{p}\left(\mathfrak{D}_{p}^{a}(\gamma)_{b}\right)=1$. In fact, we know that there is $c<b$ such that $\mathfrak{D}_{p}^{a}(\gamma)+c \in p \mathbb{Z}_{(p)}$ because $v_{p}\left(\mathfrak{D}_{p}^{a}(\gamma)_{b}\right) \geqslant 1$. So, $\mathfrak{D}_{p}^{a}(\gamma)+c=p \mathfrak{D}_{p}^{a+1}(\gamma)$ given that $c<b \leqslant p$. Therefore,

$$
\mathfrak{D}_{p}^{a}(\gamma)_{b}=p \mathfrak{D}_{p}^{a+1}(\gamma) \prod_{t=0, t \neq c}^{b-1}\left(\mathfrak{D}_{p}^{a}(\gamma)+t\right)
$$

But the $p$-adic valuation of $\prod_{t=0, t \neq c}^{b-1}\left(\mathfrak{D}_{p}^{a}(\gamma)+t\right)$ is zero because $c$ is the unique element in $\{0, \ldots, p-1\}$ such that $\mathfrak{D}_{p}^{a}(\gamma)+c \in p \mathbb{Z}_{(p)}$. Hence, $v_{p}\left(\mathfrak{D}_{p}^{a}(\gamma)_{b}\right)=1+v_{p}\left(\mathfrak{D}_{p}^{a+1}(\gamma)\right)$. Now, we show that $v_{p}\left(\mathfrak{D}_{p}^{a+1}(\gamma)\right)=0$. From the definition of $\mathfrak{D}_{p}$, it is clear that for all $t \geqslant 1$, $d_{\mathfrak{D}_{p}^{t}(\boldsymbol{\alpha}), \mathfrak{D}_{p}^{t}(\boldsymbol{\beta})}$ divides $d_{\boldsymbol{\alpha}, \boldsymbol{\beta}}$. Since, by assumption, $p>d_{\boldsymbol{\alpha}, \boldsymbol{\beta}}$, we get that, for all $t \geqslant 1$, $p>d_{\mathfrak{D}_{p}^{t}(\boldsymbol{\alpha}), \mathfrak{D}_{p}^{t}(\boldsymbol{\beta})}$. Further, $\mathfrak{D}_{p}(\gamma) \in \mathbb{Z}_{(p)}^{*} \cap(0,1]$ because $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ satisfies the $\mathbf{P}_{p, l}$ property. Thus, by (3) of Proposition 3.3, we deduce that, for all $t \geqslant 1, \mathfrak{D}_{p}^{t}(\gamma) \in \mathbb{Z}_{(p)}^{*}$. In particular, $v_{p}\left(\mathfrak{D}_{p}^{a+1}(\gamma)\right)=0$. Hence, $v_{p}\left(\mathfrak{D}_{p}^{a}(\gamma)_{b}\right)=1$.

Step III: We now prove that, for all $(a, b) \in \mathbb{Z}_{\geqslant 0} \times\{0, \ldots, p\}$,

$$
v_{p}\left(\mathcal{Q}_{\mathfrak{D}_{p}^{a}(\boldsymbol{\alpha}), \mathfrak{D}_{p}^{a}(\boldsymbol{\beta})}(b)\right)=\# \mathcal{P}_{\mathfrak{D}_{p}^{a}(\boldsymbol{\alpha}), b}-\# \mathcal{P}_{\mathfrak{D}_{p}^{a}(\boldsymbol{\beta}), b}
$$

To this end, it is sufficient to show that, for all $(a, b) \in \mathbb{Z}_{\geqslant 0} \times\{0, \ldots, p\}$,

$$
v_{p}\left(\mathfrak{D}_{p}^{a}\left(\beta_{1}\right)_{b} \cdots \mathfrak{D}_{p}^{a}\left(\beta_{n}\right)_{b}\right)=\# \mathcal{P}_{\mathfrak{D}_{p}^{a}(\boldsymbol{\beta}), b} \quad \text { and } \quad v_{p}\left(\mathfrak{D}_{p}^{a}\left(\alpha_{1}\right)_{b} \cdots \mathfrak{D}_{p}^{a}\left(\alpha_{n}\right)_{b}\right)=\# \mathcal{P}_{\mathfrak{D}_{p}^{a}(\boldsymbol{\alpha}), b}
$$

Let $i$ be in $\{0, \ldots, n\}$ such that $v_{p}\left(\mathfrak{D}_{p}^{a}\left(\beta_{i}\right)_{b}\right) \geqslant 1$. Then, according to Step II, $v_{p}\left(\mathfrak{D}_{p}^{a}\left(\beta_{i}\right)_{b}\right)=1$. Therefore, we get $v_{p}\left(\mathfrak{D}_{p}^{a}\left(\beta_{1}\right)_{r} \cdots \mathfrak{D}_{p}^{a}\left(\beta_{n}\right)_{b}\right)=\# \mathcal{P}_{\mathfrak{D}_{p}^{a}(\boldsymbol{\beta}), b}$. Similarly, we also have the equality $v_{p}\left(\mathfrak{D}_{p}^{a}\left(\alpha_{1}\right)_{b} \cdots \mathfrak{D}_{p}^{a}\left(\alpha_{n}\right)_{b}\right)=\# \mathcal{P}_{\mathfrak{D}_{p}^{a}(\boldsymbol{\alpha}), b}$.

Step IV: In this step we prove that, for all $(a, b) \in \mathbb{Z}_{\geqslant 0} \times\{0, \ldots, p\}, \# \mathcal{P}_{\mathfrak{D}_{p}^{a}(\boldsymbol{\beta}), b} \leqslant$ $\# \mathcal{P}_{\mathfrak{D}_{p}^{a}(\boldsymbol{\alpha}), b}$. By assumption ${ }_{n} F_{n-1}(\boldsymbol{\alpha}, \boldsymbol{\beta}) \in \mathbb{Z}_{(p)}[[z]]$. So, by Lemma 8.3, we deduce that, for all $(a, b) \in \mathbb{Z}_{\geqslant 0} \times\{0, \ldots, p\}, \mathcal{Q}_{\mathfrak{D}_{p}^{a}(\boldsymbol{\alpha}), \mathfrak{D}_{p}^{a}(\boldsymbol{\beta})}(b) \in \mathbb{Z}_{(p)}$. Thus, from Step III, we conclude that $\# \mathcal{P}_{\mathfrak{D}_{p}^{a}(\boldsymbol{\beta}), b} \leqslant \# \mathcal{P}_{\mathfrak{D}_{p}^{a}(\boldsymbol{\alpha}), b}$. As a consequence, we get that, for any $(a, b) \in \mathbb{Z}_{\geqslant 0} \times\{0, \ldots, p\}$, there is an injective map $\xi_{a, b}: \mathcal{P}_{\mathfrak{D}_{p}^{a}(\boldsymbol{\beta}), b} \rightarrow \mathcal{P}_{\mathfrak{D}_{p}^{a}(\boldsymbol{\alpha}), b}$. So, without losing any generality, we assume $\mathcal{P}_{\mathfrak{D}_{p}^{a}(\boldsymbol{\beta}), b} \subset \mathcal{P}_{\mathfrak{D}_{p}^{a}(\boldsymbol{\alpha}), b}$.

Step V Let $j \geqslant 1$ be an integer. We now prove that $v_{p}\left(\mathcal{Q}_{\boldsymbol{\alpha}, \boldsymbol{\beta}}(j p+r)\right)>0$. The $p$-adic expansion of $j p+r$ is of the form $j_{0}+j_{1} p+\cdots+j_{k} p^{k}$, where $j_{0}=r$ and $j_{s} \in\{0, \ldots, p-1\}$ for all $1 \leqslant s \leqslant k$. For all $0 \leqslant s<k$, we set $\tau_{s}=j_{s+1} p+\cdots+j_{k} p^{k-s}$ and $\tau_{k}=0$. From

Lemma 8.4, we deduce that

$$
\mathcal{Q}_{\boldsymbol{\alpha}, \boldsymbol{\beta}}(j p+r) \in \Lambda \mathbb{Z}_{(p)}^{*} \quad \text { with } \quad \Lambda=\prod_{s=0}^{k} \frac{\left.\left.\left(\mathfrak{D}_{p}^{s}\left(\alpha_{1}\right)+\tau_{s}\right)\right)_{j_{s}} \cdots\left(\mathfrak{D}_{p}^{s}\left(\alpha_{n}\right)+\tau_{s}\right)\right)_{j_{s}}}{\left.\left.\left(\mathfrak{D}_{p}^{s}\left(\beta_{1}\right)+\tau_{s}\right)\right)_{j_{s}} \cdots\left(\mathfrak{D}_{p}^{s}\left(\beta_{n}\right)+\tau_{s}\right)\right)_{j_{s}}} .
$$

Thus, in order to show that $v_{p}\left(\mathcal{Q}_{\boldsymbol{\alpha}, \boldsymbol{\beta}}(j p+r)\right)>0$, it is sufficient to see that $v_{p}(\Lambda)>0$.
For every $s$ in $\{0, \ldots, k\}$, we put $\left.\mathcal{J}_{s}=\left\{i \in\{0, \ldots, n\}: v_{p}\left(\left(\mathfrak{D}_{p}^{s}\left(\beta_{i}\right)+\tau_{s}\right)\right)_{j_{s}}\right) \geqslant 1\right\}$ and $\left.\mathcal{I}_{s}=\left\{i \in\{0, \ldots, n\}: v_{p}\left(\left(\mathfrak{D}_{p}^{s}\left(\alpha_{i}\right)+\tau_{s}\right)\right)_{j_{s}}\right) \geqslant 1\right\}$. Actually, $\mathcal{J}_{s}=\mathcal{P}_{\mathfrak{D}_{p}^{s}(\boldsymbol{\beta}), j_{s}}$. In fact, if $i \in \mathcal{J}_{s}$ then there exists $k_{s}<j_{s}$ such that $\mathfrak{D}_{p}^{s}\left(\beta_{i}\right)+\tau_{s}+k_{s} \in p \mathbb{Z}_{p}$. Since $\tau_{s} \in p \mathbb{Z}_{(p)}$, we have $\mathfrak{D}_{p}^{s}\left(\beta_{i}\right)+k_{s} \in \mathbb{Z}_{(p)}$. Thus, $i \in \mathcal{P}_{\mathfrak{D}_{p}^{s}(\boldsymbol{\beta}), j_{s}}$. So $\mathcal{J}_{s} \subset \mathcal{P}_{\mathfrak{D}_{p}^{s}(\boldsymbol{\beta}), j_{s}}$. In a similar way, one obtains $\mathcal{P}_{\mathfrak{D}_{p}^{s}(\boldsymbol{\beta}), j_{s}} \subset \mathcal{J}_{s}$. Similarly, one shows that $\mathcal{I}_{s}=\mathcal{P}_{\mathfrak{D}_{p}^{s}(\boldsymbol{\alpha}), j_{s}}$. Thus, from Step IV, $\mathcal{J}_{s} \subset \mathcal{I}_{s}$. In addition, $\mathcal{J}_{s}=\mathcal{J}_{s, 1} \cup \mathcal{J}_{s,>1}$, where $\mathcal{J}_{s, 1}$ is the set of $i \in \mathcal{J}_{s}$ such that $\left.v_{p}\left(\left(\mathfrak{D}_{p}^{s}\left(\beta_{i}\right)+\tau_{s}\right)\right)_{j_{s}}\right)=1$ and $\mathcal{J}_{s,>1}$ is the complement of $\mathcal{J}_{s, 1}$ in $\mathcal{J}_{s}$.

It is easily checked that $\Lambda=\Psi \Theta$, where

$$
\begin{aligned}
\Psi= & \prod_{\substack{s=0 \\
\mathcal{J}_{s,>1}=\emptyset}}^{k} \frac{\prod_{i \in \mathcal{J}_{s}}\left(\mathfrak{D}_{p}^{s}\left(\alpha_{i}\right)+\tau_{s}\right)_{j_{s}}}{\prod_{i \in \mathcal{J}_{s}}\left(\mathfrak{D}_{p}^{s}\left(\beta_{i}\right)+\tau_{s}\right)_{j_{s}}} \cdot \frac{\prod_{i \notin \mathcal{I}_{s}}\left(\mathfrak{D}_{p}^{s}\left(\alpha_{i}\right)+\tau_{s}\right)_{j_{s}}}{\prod_{i \notin \mathcal{J}_{s}}\left(\mathfrak{D}_{p}^{s}\left(\beta_{i}\right)+\tau_{s}\right)_{j_{s}}} \\
& \prod_{\substack{s=0 \\
\mathcal{J}_{s,>1} \neq \emptyset}}^{k} \frac{\prod_{i \in \mathcal{J}_{s}}\left(\mathfrak{D}_{p}^{s}\left(\alpha_{i}\right)+\tau_{s}\right)_{j_{s}}}{\prod_{i \in \mathcal{J}_{s, 1}}\left(\mathfrak{D}_{p}^{s}\left(\beta_{i}\right)+\tau_{s}\right)_{j_{s}}} \cdot \frac{\prod_{i \notin \mathcal{I}_{s}}\left(\mathfrak{D}_{p}^{s}\left(\alpha_{i}\right)+\tau_{s}\right)_{j_{s}}}{\prod_{i \notin \mathcal{J}_{s}}\left(\mathfrak{D}_{p}^{s}\left(\beta_{i}\right)+\tau_{s}\right)_{j_{s}}}
\end{aligned}
$$

and

$$
\Theta=\prod_{\substack{s=0 \\ \mathcal{J}_{s,>1}=\emptyset}}^{k} \prod_{i \in \mathcal{I}_{s} \backslash \mathcal{J}_{s}}\left(\mathfrak{D}_{p}^{s}\left(\alpha_{i}\right)+\tau_{s}\right)_{j_{s}} \cdot \prod_{\substack{s=0 \\ \mathcal{J}_{s,>1} \neq \emptyset}}^{k} \frac{\prod_{i \in \mathcal{I}_{s} \backslash \mathcal{J}_{s}}\left(\mathfrak{D}_{p}^{s}\left(\alpha_{i}\right)+\tau_{s}\right)_{j_{s}}}{\prod_{p}\left(\mathfrak{D}_{p}^{s}\left(\beta_{i}\right)+\tau_{s}\right)_{j_{s}}}
$$

We also have $\Theta=\Theta_{0} \Theta_{1}$, where

$$
\Theta_{0}=\prod_{i \in \mathcal{I}_{0} \backslash \mathcal{J}_{0}}\left(\mathfrak{D}_{p}^{s}\left(\alpha_{i}\right)+\tau_{0}\right)_{j_{0}} \quad \text { and } \quad \Theta_{1}=\frac{\prod_{s=1} \prod_{i \in \mathcal{I}_{s} \backslash \mathcal{J}_{s}}\left(\mathfrak{D}_{p}^{s}\left(\alpha_{i}\right)+\tau_{s}\right)_{j_{s}}}{\prod_{\substack{s=0 \\ \mathcal{J}_{s,>1} \neq \emptyset}}^{k} \prod_{\mathcal{J}_{s,>1}}\left(\mathfrak{D}_{p}^{s}\left(\beta_{i}\right)+\tau_{s}\right)_{j_{s}}} .
$$

We now prove $v_{p}\left(\Psi \Theta_{1}\right) \geqslant 0$. From the definition of $\Psi$, it is clear that

$$
v_{p}(\Psi) \geqslant \sum_{\substack{s=0 \\ \mathcal{J}_{s,>1} \neq \emptyset}}^{k}\left(\# \mathcal{J}_{s}-\# \mathcal{J}_{s, 1}\right)
$$

So, in order to prove that $v_{p}\left(\Psi \Theta_{1}\right) \geqslant 0$, it is sufficient to prove that

$$
\begin{equation*}
v_{p}\left(\Theta_{1}\right) \geqslant \sum_{\substack{s=0 \\ \mathcal{J}_{s,>1} \neq \emptyset}}^{k}\left(\# \mathcal{J}_{s, 1}-\# \mathcal{J}_{s}\right) \tag{8.6}
\end{equation*}
$$

Let $s$ be in $\{0, \ldots, k\}$ such that $J_{s,>1} \neq \emptyset$. Note that $k-s>0$ because $J_{k,>1}=\emptyset^{(4)}$. Let $i \in J_{s,>1}$ and let $\left.l=v_{p}\left(\left(\mathfrak{D}_{p}^{s}\left(\beta_{i}\right)+\tau_{s}\right)\right)_{j_{s}}\right)$. Then $l \geqslant 2$ and there exists $k_{s}<j_{s}$ such that $\mathfrak{D}_{p}^{s}\left(\beta_{i}\right)+\tau_{s}+k_{s} \in p \mathbb{Z}_{(p)}$. We now proceed to prove some properties which are crucial to prove Equation (8.6).
(A). We prove that $v_{p}\left(\mathfrak{D}_{p}^{s}\left(\beta_{i}\right)+\tau_{s}+k_{s}\right)=l$. In fact, we have

$$
\left.\left(\mathfrak{D}_{p}^{s}\left(\beta_{i}\right)+\tau_{s}\right)\right)_{j_{s}}=\left(\mathfrak{D}_{p}^{s}\left(\beta_{i}\right)+\tau_{s}+k_{s}\right) \prod_{t=0, t \neq k_{s}}^{j_{s}-1}\left(\mathfrak{D}_{p}^{a}\left(\beta_{i}\right)+\tau_{s}+t\right)
$$

But the $p$-adic valuation of $\prod_{t=0, t \neq k_{s}}^{j_{s}-1}\left(\mathfrak{D}_{p}^{a}\left(\beta_{i}\right)+\tau_{s}+t\right)$ is zero because $k_{s}<j_{s}<p$ and $k_{s}$ is the unique element in $\{0, \ldots, p-1\}$ such that $\mathfrak{D}_{p}^{s}\left(\beta_{i}\right)+\tau_{s}+k_{s} \in p \mathbb{Z}_{(p)}$. Thus, $v_{p}\left(\mathfrak{D}_{p}^{s}\left(\beta_{i}\right)+\right.$ $\left.\tau_{s}+k_{s}\right)=l$. In particular, we have $\mathfrak{D}_{p}^{s}\left(\beta_{i}\right)+\tau_{s}+k_{s}=p^{l} \mu$, with $\mu \in \mathbb{Z}_{(p)}^{*}$.
(B). We now show that, for all $1 \leqslant m \leqslant \min \{k-s, l-1\}, \mathfrak{D}_{p}^{s+m}\left(\beta_{i}\right)+j_{s+m} \in p \mathbb{Z}_{(p)}$ and that

$$
\mathfrak{D}_{p}^{s+m}\left(\beta_{i}\right)+j_{s+m}+j_{s+m+1} p+\cdots+j_{k} p^{k-s-m}=p^{l-m} \mu
$$

We proceed by induction on $m \in\{1, \ldots, q-1\}$, where $q=\min \{k-s, l-1\}$. From (A), we have $\mathfrak{D}_{p}^{s}\left(\beta_{i}\right)+\tau_{s}+k_{s}=p^{l} \mu$, with $\mu \in \mathbb{Z}_{(p)}^{*}$. As $\tau_{s} \in p \mathbb{Z}$ then $\mathfrak{D}_{p}^{s}\left(\beta_{i}\right)+k_{s} \in p \mathbb{Z}_{(p)}$. Therefore, $\mathfrak{D}_{p}^{s}\left(\beta_{i}\right)+k_{s}=p \mathfrak{D}_{p}^{s+1}\left(\beta_{i}\right)$ because $k_{s}<j_{s}<p$. Whence, $p \mathfrak{D}_{p}^{s+1}\left(\beta_{i}\right)+\tau_{s}=p^{l} \mu$. Remember that $\tau_{s}=j_{s+1} p+\cdots+k_{k} p^{k-s}$. Hence, $\mathfrak{D}_{p}^{s+1}\left(\beta_{i}\right)+j_{s+1}+j_{s+2} p+\cdots+j_{k} p^{k-s-1}=$ $p^{l-1} \mu$ and $\mathfrak{D}_{p}^{s+1}\left(\beta_{i}\right)+j_{s+1} \in p \mathbb{Z}_{(p)}$. We now suppose that for some $m \in\{1, \ldots, q-2\}$, $\mathfrak{D}_{p}^{s+m}\left(\beta_{i}\right)+j_{s+m} \in p \mathbb{Z}_{(p)}$ and that

$$
\mathfrak{D}_{p}^{s+m}\left(\beta_{i}\right)+j_{s+m}+j_{s+m+1} p+\cdots+j_{k} p^{k-s-m}=p^{l-m} \mu .
$$

We have $\mathfrak{D}_{p}^{s+m}\left(\beta_{i}\right)+j_{s+m}=p \mathfrak{D}_{p}^{s+m+1}\left(\beta_{i}\right)$ because $j_{s+m}<p$ and, by induction hypothesis, $\mathfrak{D}_{p}^{s+m}\left(\beta_{i}\right)+j_{s+m} \in p \mathbb{Z}_{(p)}$. So

$$
\mathfrak{D}_{p}^{s+m+1}\left(\beta_{i}\right)+j_{s+m+1}+\cdots+j_{k} p^{k-s-m}-1=p^{l-m-1} \mu .
$$

(C). We now see that $l \leqslant k-s+1$. Suppose, towards a contradiction, that $l>k-s+1$. From (B), we know that, for all $m \in\{1, \ldots, k-s\}$,

$$
\mathfrak{D}_{p}^{s+m}\left(\beta_{i}\right)+j_{s+m}+j_{s+m+1} p+\cdots+j_{k} p^{k-s-m}=p^{l-m} \mu .
$$

In particular, $\mathfrak{D}_{p}^{k}\left(\beta_{i}\right)+j_{k}=p^{l-k+s} \mu$ and $\mathfrak{D}_{p}^{k}\left(\beta_{i}\right)+j_{k} \in p \mathbb{Z}_{(p)}$. Hence, $\mathfrak{D}_{p}^{k}\left(\beta_{i}\right)+j_{k}=$ $p \mathfrak{D}_{p}^{k+1}\left(\beta_{i}\right)$ because $j_{k}<p$. So $\mathfrak{D}_{p}^{k+1}\left(\beta_{i}\right)=p^{l-k+s-1} \mu$. But $l-k+s-1>0$ and hence, $\mathfrak{D}_{p}^{k+1}\left(\beta_{i}\right) \in p \mathbb{Z}_{(p)}$. From the definition of $\mathfrak{D}_{p}$, it is clear that, for all $t \geqslant 1, d_{\mathfrak{D}_{p}^{t}(\boldsymbol{\alpha}), \mathfrak{D}_{p}^{t}(\boldsymbol{\beta})}$ divides $d_{\boldsymbol{\alpha}, \boldsymbol{\beta}}$. Since by assumption, $p>d_{\boldsymbol{\alpha}, \boldsymbol{\beta}}$, we get that, for all $t \geqslant 1, p>d_{\mathfrak{D}_{p}^{t}(\boldsymbol{\alpha}), \mathfrak{D}_{p}^{t}(\boldsymbol{\beta})}$. Further, $\mathfrak{D}_{p}\left(\beta_{i}\right) \in \mathbb{Z}_{(p)}^{*} \cap(0,1]$ because $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ satisfies the $\mathbf{P}_{p, l}$ property. Thus, by (3) of Proposition 3.3, we deduce that, for all $t \geqslant 1, \mathfrak{D}_{p}^{t}\left(\beta_{i}\right) \in \mathbb{Z}_{(p)}^{*}$. In particular, $\mathfrak{D}_{p}^{k+1}\left(\beta_{i}\right) \in \mathbb{Z}_{(p)}^{*}$, which is a contradiction to the fact that $\mathfrak{D}_{p}^{k+1}\left(\beta_{i}\right) \in p \mathbb{Z}_{(p)}$. Consequently, $l \leqslant k-s+1$.
(D). Now, we prove see that, for every $m \in\{1, \ldots, l-1\}, i \in \mathcal{I}_{s+m} \backslash \mathcal{J}_{s+m}$. From (B) and $(\mathbf{C})$, we have $\mathfrak{D}_{p}^{s+m}\left(\beta_{i}\right)+j_{s+m} \in p \mathbb{Z}_{(p)}$ for all $m \in\{1, \ldots, l-1\}$. Then $i \in \mathcal{P}_{\mathfrak{D}_{p}^{s+m}(\boldsymbol{\beta}), j_{s+m}+1}$.

[^4]By Step IV, we have $\mathcal{P}_{\mathfrak{D}_{p}^{s+m}(\boldsymbol{\beta}), j_{s+m}+1} \subset \mathcal{P}_{\mathfrak{D}_{p}^{s+m}(\boldsymbol{\alpha}), j_{s+m}+1}$. Hence, $i \in \mathcal{P}_{\mathfrak{D}_{p}^{s+m}(\boldsymbol{\alpha}), j_{s+m}+1}$. From Step I, we know that, for all $a \geqslant 1, \mathfrak{D}_{p}^{a}\left(\alpha_{i}\right)-\mathfrak{D}_{p}^{a}\left(\beta_{j}\right)$ belongs to $\mathbb{Z}_{(p)}^{*}$ for all $1 \leqslant i, j \leqslant n$. Since $\mathfrak{D}_{p}^{s+m}\left(\beta_{i}\right)+j_{s+m} \in p \mathbb{Z}_{(p)}$, we get that, for all $r \in\{1, \ldots, n\}, \mathfrak{D}_{p}^{s+m}\left(\alpha_{r}\right)+j_{s+m} \notin p \mathbb{Z}_{(p)}$. Hence, $i \in \mathcal{P}_{\mathfrak{D}_{p}^{s+m}(\boldsymbol{\alpha}), j_{s+m}}=\mathcal{I}_{s+m}$. We now show that $i \notin \mathcal{J}_{s+m}$. Aiming for a contradiction, suppose that $i \in \mathcal{J}_{s+m}$. Thus, there is $d<j_{s+m}$ such that $\mathfrak{D}_{p}^{s+m}\left(\beta_{i}\right)+d+\tau_{j_{s+m}} \in p \mathbb{Z}_{p}$. Since $\tau_{j_{s+m}} \in p \mathbb{Z}, \mathfrak{D}_{p}^{s+m}\left(\beta_{i}\right)+d \in p \mathbb{Z}_{p}$. But we know that $\mathfrak{D}_{p}^{s+m}\left(\beta_{i}\right)+j_{s+m} \in p \mathbb{Z}_{(p)}$. Thus, $j_{s+m}-d \in p \mathbb{Z}$. That is a contradiction because $0 \leqslant d<j_{s+m}<p$. Consequently, for all $m \in\{1, \ldots, l-1\}, i \in \mathcal{I}_{s+m} \backslash \mathcal{J}_{s+m}$. In particular, $\left.v_{p}\left(\mathfrak{D}_{p}^{s}\left(\alpha_{i}\right)+\tau_{s+m}\right)_{j_{s+m}}\right) \geqslant 1$. Whence,

$$
v_{p}\left(\frac{\prod_{m=1}^{l-1}\left(\mathfrak{D}_{p}^{s}\left(\alpha_{i}\right)+\tau_{s+m}\right)_{j_{s+m}}}{\left(\mathfrak{D}_{p}^{s}\left(\beta_{i}\right)+\tau_{j_{s}}\right)_{j_{s}}}\right) \geqslant-1
$$

From (C), we have $l \leqslant k-s+1$. Thus, for all $m \in\{1, \ldots, l-1\}, s+m \leqslant k$. So, from (D) it follows that the product $\prod_{m=1}^{l-1}\left(\mathfrak{D}_{p}^{s}\left(\alpha_{i}\right)+\tau_{s+m}\right)_{j_{s+m}}$ is a factor of $\prod_{s=1}^{k} \prod_{i \in \mathcal{I}_{s} \backslash \mathcal{J}_{s}}\left(\mathfrak{D}_{p}^{s}\left(\alpha_{i}\right)+\tau_{s}\right)_{j_{s}}$. Consequenly,

$$
v_{p}\left(\Theta_{1}\right) \geqslant \sum_{\substack{s=0 \\ \mathcal{J}_{s,>1} \neq \emptyset}}^{k}\left(\# \mathcal{J}_{s, 1}-\# \mathcal{J}_{s}\right)
$$

because $\# \mathcal{J}_{s,>1}=\# \mathcal{J}_{s}-\# \mathcal{J}_{s, 1}$.
Finally, from Step III, we have

$$
v_{p}\left(\mathcal{Q}_{\boldsymbol{\alpha}, \boldsymbol{\beta}}(r)\right)=\# \mathcal{P}_{\boldsymbol{\alpha}, r}-\# \mathcal{P}_{\boldsymbol{\beta}, r}
$$

By assumption, $v_{p}\left(\mathcal{Q}_{\boldsymbol{\alpha}, \boldsymbol{\beta}}(r)\right)>0$. Since $r=j_{0}$, it is not hard to see that $\mathcal{J}_{0}=\mathcal{P}_{\boldsymbol{\beta}, r}$ and that $\mathcal{I}_{0}=\mathcal{P}_{\boldsymbol{\alpha}, r}$. Thus $\# \mathcal{J}_{0}<\# \mathcal{I}_{0}$. Whence, $v_{p}\left(\Theta_{0}\right)>0$. This completes the proof.

Lemma 8.6. - Let $\boldsymbol{\alpha}=\left(\alpha_{1}, \ldots, \alpha_{n}\right)$ and $\boldsymbol{\beta}=\left(\beta_{1}, \ldots, \beta_{n-1}, 1\right)$ be in $\left(\mathbb{Q} \backslash \mathbb{Z}_{\leqslant 0}\right)^{n}$ and let $p$ be a prime number such that $\mathcal{H}(\boldsymbol{\alpha}, \boldsymbol{\beta}) \in \mathbb{Z}_{(p)}[z][\delta]$. Suppose that $A(z)=a_{m} z^{m}+\cdots+a_{r} z^{r} \in$ $\mathbb{F}_{p}[z]$ is a solution of $\mathcal{H}(\boldsymbol{\alpha}, \boldsymbol{\beta}, p)$. If $a_{m} \neq 0$ then $m \bmod p$ is an exponent at zero of $\mathcal{H}(\boldsymbol{\alpha}, \boldsymbol{\beta}, p)$, that is, $m \bmod p$ belongs to $\left\{0,1-\beta_{1} \bmod p, \ldots, 1-\beta_{n-1} \bmod p\right\}$.

Proof. - It is not hard to see that

$$
\mathcal{H}(\boldsymbol{\alpha}, \boldsymbol{\beta}, p)(A(z))=\prod_{j=1}^{n}\left(m+\beta_{j}-1 \bmod p\right) a_{m} z^{m}+z^{m+1} B(z)
$$

where $B(z)$ is a polynomial. As $\mathcal{H}(\boldsymbol{\alpha}, \boldsymbol{\beta}, p)(A(z))=0$ then $\prod_{j=1}^{n}\left(m+\beta_{j}-1 \bmod p\right) a_{m}=0$. So, $\prod_{j=1}^{n}\left(m+\beta_{j}-1 \bmod p\right)=0$ because, by hypothesis, $a_{m} \neq 0$. Thus, we have $m \bmod p=$ $1-\beta_{j} \bmod p$ for some $j \in\{1, \ldots, n\}$.

### 8.3. Proof of Lemma 8.2

Let us write $E_{\boldsymbol{\alpha}, \boldsymbol{\beta}, p}=\left\{e_{0}, e_{1}, \ldots, e_{k}\right\}$, where $e_{i}<e_{i+1}$ for all $i \in\{0, \ldots, k\}$ and $e_{0}=0$. We set $e_{k+1}=p$. So, for all $e_{i} \in\left\{e_{0}, e_{1}, \ldots, e_{k}\right\}$ and, for all nonnegative intergers $j$, we set

$$
P_{j, e_{i}}(z)=\sum_{s=j p+e_{i}}^{j p+e_{i+1}-1}\left(\mathcal{Q}_{\boldsymbol{\alpha}, \boldsymbol{\beta}}(s) \bmod p\right) z^{s}
$$

We split the proof into three steps:
Step I. For all $e_{i} \in\left\{e_{0}, \ldots, e_{k}\right\}$ and for all nonnegative integers $j$, the polynomial $P_{j, e_{i}}(z)$ is a solution of $\mathcal{H}(\boldsymbol{\alpha}, \boldsymbol{\beta}, p)$.

Step II. If $\mathcal{Q}_{\boldsymbol{\alpha}, \boldsymbol{\beta}}\left(e_{i}\right) \bmod p=0$ then, for every integer $j \geqslant 0, P_{j, e_{i}}(z)$ is the zero polynomial.

Step III. If $\mathcal{Q}_{\boldsymbol{\alpha}, \boldsymbol{\beta}}\left(e_{i}\right) \bmod p \neq 0$ then, for every integer $j \geqslant 0$,

$$
P_{j, e_{i}}(z)=\frac{\mathcal{Q}_{\boldsymbol{\alpha}, \boldsymbol{\beta}}\left(j p+e_{i}\right)}{\mathcal{Q}_{\boldsymbol{\alpha}, \boldsymbol{\beta}}\left(e_{i}\right)} \bmod p \cdot z^{j p} P_{0, e_{i}}(z)
$$

Proof of Step I. - It is not hard to see that

$$
\begin{equation*}
\mathcal{H}(\boldsymbol{\alpha}, \boldsymbol{\beta})=(1-z) \delta^{n}+\left[S_{n, 1}(\boldsymbol{\beta}-\mathbf{1})-z S_{n, 1}(\boldsymbol{\alpha})\right] \delta^{n-1}+\cdots+S_{n, n}(\boldsymbol{\beta}-\mathbf{1})-z S_{n, n}(\boldsymbol{\alpha}) \tag{8.7}
\end{equation*}
$$

where $\mathbf{1}=(1, \ldots, 1) \in \mathbb{N}^{n}$ and $S_{n, r}=\sum_{1 \leqslant i_{1}<\cdots<i_{r} \leqslant n} X_{i_{1}} \cdots X_{i_{r}}$.
Now, we set $I(z)=\prod_{i=1}^{n}\left(z+\beta_{i}-1\right)$ and $T(z)=\prod_{i=1}^{n}\left(z+\alpha_{i}\right)$. Then, it follows from Equality (8.7) that

$$
\begin{aligned}
\mathcal{H}(\boldsymbol{\alpha}, \boldsymbol{\beta}, p)\left(P_{j, e_{i}}(z)\right) & =\left(I\left(j p+e_{i}\right) \mathcal{Q}_{\boldsymbol{\alpha}, \boldsymbol{\beta}}\left(j p+e_{i}\right) \bmod p\right) z^{j p+e_{i}} \\
& +\sum_{k=j p+e_{i}+1}^{j p+e_{i+1}-1}\left(\left(I(k) \mathcal{Q}_{\boldsymbol{\alpha}, \boldsymbol{\beta}}(k)-T(k-1) \mathcal{Q}_{\boldsymbol{\alpha}, \boldsymbol{\beta}}(k-1)\right) \bmod p\right) z^{k} \\
& -\left(T\left(j p+e_{i+1}-1\right) \mathcal{Q}_{\boldsymbol{\alpha}, \boldsymbol{\beta}}\left(j p+e_{i+1}-1\right) \bmod p\right) z^{j p+e_{i+1}}
\end{aligned}
$$

We now prove that $\mathcal{H}(\boldsymbol{\alpha}, \boldsymbol{\beta}, p)\left(P_{j, e_{i}}(z)\right)=0$. Recall that, by definition, $e_{i} \bmod p$ belongs to $\left\{0,1-\beta_{1}, \ldots, 1-\beta_{n-1}\right\} \bmod p$. Then, we have $I\left(e_{i}\right) \bmod p \equiv 0$. But, $I\left(j p+e_{i}\right) \equiv$ $I\left(e_{i}\right) \bmod p$.Thus, $I\left(j p+e_{i}\right) \bmod p \equiv 0$. So that, $I\left(j p+e_{i}\right) \mathcal{Q}_{\boldsymbol{\alpha}, \boldsymbol{\beta}}\left(j p+e_{i}\right) \bmod p \equiv 0$. Furthermore, it is clear that, for every positive integer $t$ we have

$$
I(t) \mathcal{Q}_{\boldsymbol{\alpha}, \boldsymbol{\beta}}(t)-\mathcal{Q}_{\boldsymbol{\alpha}, \boldsymbol{\beta}}(t-1) T(t-1)=0
$$

By hypotheses, $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ belong to $\mathbb{Z}_{(p)}^{n}$. Thus, for every integer $t, I(t)$ and $T(t-1)$ belong to $\mathbb{Z}_{(p)}$. Again, by hypotheses, we know that ${ }_{n} F_{n-1}(\boldsymbol{\alpha}, \boldsymbol{\beta} ; z)$ belongs to $\mathbb{Z}_{(p)}[[z]]$. Consequently, for every integer $t \geqslant 1, \mathcal{Q}_{\boldsymbol{\alpha}, \boldsymbol{\beta}}(t)$ and $\mathcal{Q}_{\boldsymbol{\alpha}, \boldsymbol{\beta}}(t-1)$ belong to $\mathbb{Z}_{(p)}$. Therefore, from the previous equality we conclude that, for every positive integer $t$,

$$
\begin{equation*}
\left(I(t) \mathcal{Q}_{\boldsymbol{\alpha}, \boldsymbol{\beta}}(t)-\mathcal{Q}_{\boldsymbol{\alpha}, \boldsymbol{\beta}}(t-1) T(t-1)\right) \bmod p=0 \tag{8.8}
\end{equation*}
$$

In particular, for every $k \in\left\{j p+e_{i}+1, \ldots, j p+e_{i+1}-1\right\}$, we have

$$
\left(I(k) \mathcal{Q}_{\boldsymbol{\alpha}, \boldsymbol{\beta}}(k)-T(k-1) \mathcal{Q}_{\boldsymbol{\alpha}, \boldsymbol{\beta}}(k-1)\right) \bmod p=0
$$

and

$$
\left(I\left(j p+e_{i+1}\right) \mathcal{Q}_{\boldsymbol{\alpha}, \boldsymbol{\beta}}\left(j p+e_{i+1}\right)-\mathcal{Q}_{\boldsymbol{\alpha}, \boldsymbol{\beta}}\left(j p+e_{i+1}-1\right) T\left(j p+e_{i+1}-1\right)\right) \bmod p=0
$$

Since $e_{i+1} \bmod p$ is an exponent at zero of $\mathcal{H}(\boldsymbol{\alpha}, \boldsymbol{\beta}, p), I\left(e_{i+1}\right) \bmod p \equiv 0$. But, it is clear that $I\left(j p+e_{i+1}\right) \equiv I\left(e_{i+1}\right) \bmod p$. Thus, $I\left(j p+e_{i+1}\right) \bmod p \equiv 0$. Therefore, we have

$$
\left.\mathcal{Q}_{\boldsymbol{\alpha}, \boldsymbol{\beta}}\left(j p+e_{i+1}-1\right) T\left(j p+e_{i+1}-1\right)\right) \equiv 0 \bmod p
$$

So that, $\mathcal{H}(\boldsymbol{\alpha}, \boldsymbol{\beta}, p)\left(P_{j, e_{i}}(z)\right)=0$.

Proof of Step II. - Suppose that $\mathcal{Q}_{\boldsymbol{\alpha}, \boldsymbol{\beta}}\left(e_{i}\right) \bmod p=0$. Let $j$ be a nonnegative integer. We want to show that $P_{j, e_{i}}(z)$ is the zero polynomial. For this purpose, we show by induction on $s \in\left\{j p+e_{i}, j p+e_{i}+1, \ldots, j p+e_{i+1}-1\right\}$ that $\mathcal{Q}_{\boldsymbol{\alpha}, \boldsymbol{\beta}}(s) \equiv 0 \bmod p$. Since $v_{p}\left(\mathcal{Q}_{\boldsymbol{\alpha}, \boldsymbol{\beta}}\left(e_{i}\right)\right)>0$, by Lemma 8.5, we have $v_{p}\left(\mathcal{Q}_{\boldsymbol{\alpha}, \boldsymbol{\beta}}\left(j p+e_{i}\right)\right)>0$. So that, $\left.\mathcal{Q}_{\boldsymbol{\alpha}, \boldsymbol{\beta}}\left(j p+e_{i}\right)\right) \bmod p=0$. Now, suppose that $\mathcal{Q}_{\boldsymbol{\alpha}, \boldsymbol{\beta}}(s) \equiv 0 \bmod p$ for some $s$ in the set $\left\{j p+e_{i}, j p+e_{i}+1, \ldots, j p+e_{i+1}-2\right\}$. From Equation (8.8) we know that $\left(I(s+1) \mathcal{Q}_{\boldsymbol{\alpha}, \boldsymbol{\beta}}(s+1)-\mathcal{Q}_{\boldsymbol{\alpha}, \boldsymbol{\beta}}(s) T(s)\right) \bmod p=0$. By applying our induction hypothesis, we obtain $I(s+1) \mathcal{Q}_{\boldsymbol{\alpha}, \boldsymbol{\beta}}(s+1) \bmod p=0$. Suppose, towards a contradiction, that $I(s+1) \equiv 0 \bmod p$. Then, $s+1 \bmod p$ is an exponent at zero of $\mathcal{H}(\boldsymbol{\alpha}, \boldsymbol{\beta}, p)$ and since $j p+e_{i}<s+1<j p+e_{i+1}$, we have $e_{i}<s+1-j p<e_{i+1}$. So, $0 \leqslant s+1-j p<p$ and therefore, $s+1-j p \in E_{\boldsymbol{\alpha}, \boldsymbol{\beta}, p}$. Hence, there is $m \in\{0, \ldots, k\}$ such that $e_{m}=s+1-j p$. Then, $e_{i}<e_{m}<e_{i+1}$. Now, we know that if $i<m$ then $e_{i+1} \leqslant e_{m}$ and if $m<i$ then $e_{m}<e_{i}$. This is a clear contradiction of the fact that $e_{i}<e_{m}<e_{i+1}$. Thus, $I(s+1) \bmod p \neq 0$. Then, it follows that $\mathcal{Q}_{\boldsymbol{\alpha}, \boldsymbol{\beta}}(s+1) \bmod p=0$. Therefore, $P_{j, e_{i}}(z)$ is the zero polynomial.

Proof of Step III. - From Step I, we know that $P_{j, e_{i}}$ and $P_{0, e_{i}}$ are solutions of $\mathcal{H}(\boldsymbol{\alpha}, \boldsymbol{\beta}, p)$. Thus, the polynomial $P_{j, e_{i}}(z)-\frac{\mathcal{Q}_{\alpha, \boldsymbol{\beta}}\left(j p+e_{i}\right)}{\mathcal{Q}_{\alpha, \boldsymbol{\beta}}\left(e_{i}\right)} \bmod p \cdot z^{j p} P_{0, e_{i}}(z)$ is a solution of $\mathcal{H}(\boldsymbol{\alpha}, \boldsymbol{\beta}, p)$. Suppose, to derive a contradiction, that this polynomial is not zero. Then, from Lemma 8.6 it follows that the differential operator $\mathcal{H}(\boldsymbol{\alpha}, \boldsymbol{\beta}, p)$ has an exponent at zero in the set $\left\{\left(e_{i}+\right.\right.$ 1) $\left.\bmod p, \ldots,\left(e_{i+1}-1\right) \bmod p\right\}$ because

$$
\begin{aligned}
& P_{j, e_{i}}(z)-\frac{\mathcal{Q}_{\boldsymbol{\alpha}, \boldsymbol{\beta}}\left(j p+e_{i}\right)}{\mathcal{Q}_{\boldsymbol{\alpha}, \boldsymbol{\beta}}\left(e_{i}\right)} \bmod p \cdot z^{j p} P_{0, e_{i}}(z) \\
& =\sum_{s=e_{i}+1}^{e_{i+1}-1}\left(\mathcal{Q}_{\boldsymbol{\alpha}, \boldsymbol{\beta}}(j p+s)-\frac{\mathcal{Q}_{\boldsymbol{\alpha}, \boldsymbol{\beta}}\left(j p+e_{i}\right) \mathcal{Q}_{\boldsymbol{\alpha}, \boldsymbol{\beta}}(s)}{\mathcal{Q}_{\boldsymbol{\alpha}, \boldsymbol{\beta}}\left(e_{i}\right)} \bmod p\right) z^{j p+s} .
\end{aligned}
$$

Therefore, there exits $e_{m} \in E_{\boldsymbol{\alpha}, \beta, p} \cap\left\{e_{i}+1, \ldots, e_{i+1}-1\right\}$. If $i<m$ then $e_{i+1} \leqslant e_{m}$ and if $m<i$ then $e_{m}<e_{i}$. This leads to the contradiction that $e_{m} \in\left\{e_{i}+1, \ldots, e_{i+1}-1\right\}$.

Consequently, the polynomial $P_{j, e_{i}}(z)-\frac{\mathcal{Q}_{\alpha, \boldsymbol{\beta}}\left(j p+e_{i}\right)}{\mathcal{Q}_{\alpha, \boldsymbol{\beta}}\left(e_{i}\right)} \bmod p \cdot z^{j p} P_{0, e_{i}}(z)$ is the zero polynomial.

Now, we are in a position to finish the proof of the lemma. Let us write $S_{\boldsymbol{\alpha}, \boldsymbol{\beta}, p}=$ $\left\{0, r_{1}, \ldots, r_{t}\right\}$. It is clear that

$$
f(z) \bmod p=\sum_{j \geqslant 0}\left(\sum_{i=0}^{k} P_{j, e_{i}}(z)\right) .
$$

From Step II we deduce that if $e_{i} \in E_{\boldsymbol{\alpha}, \boldsymbol{\beta}, p} \backslash S_{\boldsymbol{\alpha}, \boldsymbol{\beta}, p}$ then, for all integers $j \geqslant 0, P_{j, e_{i}}(z)$ is the zero polynomial. Then, we have

$$
f(z) \bmod p=\sum_{j \geqslant 0} P_{j, 0}(z)+\sum_{j \geqslant 0} P_{j, r_{1}}(z)+\cdots+\sum_{j \geqslant 0} P_{j, r_{t}}(z) .
$$

From Step III we conclude that if $r_{i} \in S_{\boldsymbol{\alpha}, \boldsymbol{\beta}, p}$ then, for all integers $j \geqslant 0$,

$$
P_{j, r_{i}}(z)=\mathcal{Q}_{\boldsymbol{\alpha}, \boldsymbol{\beta}}\left(j p+r_{i}\right) \bmod p \cdot z^{j p} P_{i}(z),
$$

where $P_{i}(z)=\sum_{s=r_{i}}^{e_{j}-1} \frac{\mathcal{Q}_{\alpha, \mathcal{\beta}}(s)}{\mathcal{Q}_{\alpha, \boldsymbol{\beta}}\left(r_{i}\right)} z^{s}$ with $e_{j-1}=r_{i}$.
Therefore,

$$
\begin{aligned}
\sum_{j \geqslant 0} P_{j, r_{i}}(z) & =\sum_{j \geqslant 0}\left(\mathcal{Q}_{\boldsymbol{\alpha}, \boldsymbol{\beta}}\left(j p+r_{i}\right) \bmod p \cdot z^{j p} P_{i}(z)\right) \\
& =P_{i}(z) \sum_{j \geqslant 0}\left(\mathcal{Q}_{\boldsymbol{\alpha}, \boldsymbol{\beta}}\left(j p+r_{i}\right) \bmod p \cdot z^{j p}\right) \\
& =P_{i}(z) \Lambda_{r_{i}}(f)^{p} .
\end{aligned}
$$

Consequently, we have

$$
f(z) \equiv \sum_{r \in S_{\boldsymbol{\alpha}, \boldsymbol{\beta}, p}} P_{r}(z) \Lambda_{r}(f)^{p} \bmod p \quad \text { with } \quad P_{r}(z)=\sum_{s=r}^{r^{\prime}-1} \frac{\mathcal{Q}_{\boldsymbol{\alpha}, \boldsymbol{\beta}}(s)}{\mathcal{Q}_{\boldsymbol{\alpha}, \boldsymbol{\beta}}(r)} z^{s}
$$

### 8.4. Auxiliary result II

With the aim of carrying out the proof of Lemma 8.1, we need one more auxiliary result.
Lemma 8.7. - Let $p$ be a primer number, let $r$ be in $\{0,1, \ldots, p-1\}$ and let $\boldsymbol{\alpha}=$ $\left(\alpha_{1}, \ldots, \alpha_{n}\right)$ and $\boldsymbol{\beta}=\left(\beta_{1}, \ldots, \beta_{n-1}, 1\right)$ be in $\left(\mathbb{Z}_{(p)} \backslash \mathbb{Z}_{\leqslant 0}\right)^{n}$. Consider the following elements:
with $j \in \mathbb{Z}_{\geqslant 0}$. Suppose that $\mathfrak{D}_{p}(\boldsymbol{\alpha}), \mathfrak{D}_{p}(\boldsymbol{\beta})$ belong to $\left(\mathbb{Z}_{(p)}^{*}\right)^{n}$ and that $v_{p}\left(\mathcal{Q}_{\boldsymbol{\alpha}, \boldsymbol{\beta}}(r)\right)=0$. Then:
(1) $v_{p}\left(\lambda_{0}\right)=v_{p}(\theta)=v_{p}(\tau)=0$,
(2) $\# \mathcal{P}_{\boldsymbol{\alpha}, r}=\# \mathcal{P}_{\boldsymbol{\beta}, r}$ and $\mathcal{Q}_{\boldsymbol{\alpha}, \boldsymbol{\beta}}(r)=\lambda_{0} \tau \theta$,
(3) for every integer $j \geqslant 0, \mathcal{Q}_{\boldsymbol{\alpha}, \boldsymbol{\beta}}(j p+r)=\mathcal{Q}_{\mathfrak{D}_{p}(\boldsymbol{\alpha}), \mathfrak{D}_{p}(\boldsymbol{\beta}), p}(j) \lambda_{j} \nu$, where $\nu \in \mathbb{Z}_{(p)}^{*}$ and

$$
\nu \bmod p=(\tau \bmod p)(\theta \bmod p)
$$

(4) if for every integer $j \geqslant 0, v_{p}\left(\mathcal{Q}_{\boldsymbol{\alpha}, \boldsymbol{\beta}}(j)\right) \geqslant 0$, then, for every $j \geqslant 0$, we have $v_{p}\left(\mathcal{Q}_{\mathfrak{D}_{p}(\boldsymbol{\alpha}), \mathfrak{D}_{p}(\boldsymbol{\beta})}(j) \lambda_{j}\right) \geqslant 0$ and

$$
v_{p}\left(\frac{\prod_{s \in \mathcal{C}_{\alpha, r}} \mathfrak{D}_{p}\left(\alpha_{s}\right)_{j} \prod_{s \in \mathcal{P}_{\alpha, r}}\left(\mathfrak{D}_{p}\left(\alpha_{s}\right)+1\right)_{j}}{\prod_{s \in \mathcal{C}_{\boldsymbol{\beta}, r}} \mathfrak{D}_{p}\left(\beta_{s}\right)_{j} \prod_{s \in \mathcal{P}_{\boldsymbol{\beta}, r}}\left(\mathfrak{D}_{p}\left(\beta_{s}\right)+1\right)_{j}}\right) \geqslant 0
$$

(5) if for every integer $j \geqslant 0, v_{p}\left(\mathcal{Q}_{\boldsymbol{\alpha}, \boldsymbol{\beta}}(j)\right) \geqslant 0$, then, for every integer $j \geqslant 0$,
$\mathcal{Q}_{\boldsymbol{\alpha}, \boldsymbol{\beta}}(j p+r) \bmod p=\left(\frac{\prod_{s \in \mathcal{C}_{\alpha, r}} \mathfrak{D}_{p}\left(\alpha_{s}\right)_{j} \prod_{s \in \mathcal{P}_{\boldsymbol{\alpha}, r}}\left(\mathfrak{D}_{p}\left(\alpha_{s}\right)+1\right)_{j}}{\prod_{s \in \mathcal{C}_{\boldsymbol{\beta}, r}} \mathfrak{D}_{p}\left(\beta_{s}\right)_{j} \prod_{s \in \mathcal{P}_{\boldsymbol{\beta}, r}}\left(\mathfrak{D}_{p}\left(\beta_{s}\right)+1\right)_{j}} \bmod p\right)\left(\mathcal{Q}_{\boldsymbol{\alpha}, \boldsymbol{\beta}}(r) \bmod p\right)$.
Proof. - We first suppose that $r=0$. Then $\mathcal{P}_{\boldsymbol{\alpha}, r}=\emptyset=\mathcal{P}_{\boldsymbol{\beta}, r}$. So, $\tau=1$ and, for all $j \in \mathbb{Z}_{\geqslant 0}, \lambda_{j}=1$. Also, it is clear that $\theta=1$. Therefore, (1) and (2) are satisfied and (3), (4), and (5) follows immediately from Lemma 8.3. We now suppose that $r>0$.

- We prove that $v_{p}(\tau)=0$. If $s \in \mathcal{P}_{\boldsymbol{\alpha}, r}$ then the $p$-adic valuation of

$$
\prod_{\substack{t=0 \\ t \neq p \mathfrak{D}_{p}\left(\alpha_{s}\right)-\alpha_{s}}}^{r-1}\left(\alpha_{s}+t\right)
$$

is zero because $k=p \mathfrak{D}_{p}\left(\alpha_{s}\right)-\alpha_{s}$ is the unique element in $\{0,1, \ldots, p-1\}$ such that $\alpha_{s}+k \in p \mathbb{Z}_{(p)}$ and, by assumption, $0<r<p$. Similarly, if $s \in \mathcal{P}_{\boldsymbol{\beta}, r}$ then the $p$-adic valuation of

$$
\prod_{\substack{t=0 \\ t \neq p \mathfrak{D}_{p}\left(\beta_{s}\right)-\beta_{s}}}^{\left.\substack{r-1} \beta_{s}+t\right)}
$$

is zero. Therefore, $v_{p}(\tau)=0$.

- We prove that $v_{p}(\theta)=0$. It is clear that if $s \in \mathcal{C}_{\boldsymbol{\alpha}, r}$ then the $p$-adic valuation of $\left(\alpha_{s}\right)_{r}$ is zero. Likewise, if $s \in \mathcal{C}_{\boldsymbol{\beta}, r}$ then the $p$-adic valuation of $\left(\beta_{s}\right)_{r}$ is zero. Thus, the $p$-adic valuation of $\theta$ is zero.
- Finally, $v_{p}\left(\lambda_{0}\right)=0$ because by assumption, $\mathfrak{D}_{p}(\boldsymbol{\alpha}), \mathfrak{D}_{p}(\boldsymbol{\beta})$ belong to $\left(\mathbb{Z}_{(p)}^{*}\right)^{n}$.
(2). It is clear that $\mathcal{Q}_{\boldsymbol{\alpha}, \boldsymbol{\beta}}(r)=p^{\# \mathcal{P}_{\boldsymbol{\alpha}, r}-\# \mathcal{P}_{\boldsymbol{\beta}, r}} \lambda_{0} \tau \theta$. From (1), we know that $v_{p}(\tau)=$ $v_{p}\left(\lambda_{0}\right)=v_{p}(\theta)=0$ and by asumption, $v_{p}\left(\mathcal{Q}_{\boldsymbol{\alpha}, \boldsymbol{\beta}}(r)\right)=0$. Thus, $v_{p}\left(p^{\# \mathcal{P}_{\boldsymbol{\alpha}, r}-\# \mathcal{P}_{\boldsymbol{\beta}, r}}\right)=0$. Whence, $\# \mathcal{P}_{\boldsymbol{\alpha}, r}=\# \mathcal{P}_{\boldsymbol{\beta}, r}$. So, $\mathcal{Q}_{\boldsymbol{\alpha}, \boldsymbol{\beta}}(r)=\lambda_{0} \tau \theta$.
(3). Let $j$ be a nonnegative integer. The following equality is straightforward

$$
\mathcal{Q}_{\boldsymbol{\alpha}, \boldsymbol{\beta}}(j p+r)=\mathcal{Q}_{\boldsymbol{\alpha}, \boldsymbol{\beta}}(j p) \mathcal{Q}_{\boldsymbol{\alpha}+\mathbf{j}, \boldsymbol{p}, \mathbf{j} \mathbf{p}}(r)
$$

where

$$
\mathcal{Q}_{\boldsymbol{\alpha}+\mathbf{j} \mathbf{p}, \boldsymbol{\beta}+\mathbf{j} \mathbf{p}}(r)=\frac{\left(\alpha_{1}+j p\right)_{r} \cdots\left(\alpha_{n}+j p\right)_{r}}{\left(\beta_{1}+j p\right)_{r} \cdots\left(\beta_{n}+j p\right)_{r}}
$$

Clearly, we also have

$$
\mathcal{Q}_{\boldsymbol{\alpha}+\mathbf{j} \mathbf{p}, \boldsymbol{\beta}+\mathbf{j} \mathbf{p}}(r)=\frac{\prod_{s \in \mathcal{P}_{\boldsymbol{\alpha}, r}}\left(\alpha_{s}+j p\right)_{r} \prod_{s \in \mathcal{C}_{\boldsymbol{\alpha}, r}}\left(\alpha_{s}+j p\right)_{r}}{\prod_{s \in \mathcal{P}_{\boldsymbol{\beta}, r}}\left(\beta_{s}+j p\right)_{r} \prod_{s \in \mathcal{C}_{\boldsymbol{\beta}, r}}\left(\beta_{s}+j p\right)_{r}} .
$$

By (2), we have $\# \mathcal{P}_{\boldsymbol{\alpha}, r}=\# \mathcal{P}_{\boldsymbol{\beta}, r}$ and thus,

$$
\mathcal{Q}_{\boldsymbol{\alpha}+\mathbf{j}, \boldsymbol{\beta}+\mathbf{j} \mathbf{p}}(r)=\lambda_{j} \cdot \xi, \text { where } \quad \xi=\frac{\prod_{s \in \mathcal{P}_{\boldsymbol{\alpha}, r}}\binom{\prod_{t=0}^{r-1}\left(\alpha_{s}+j p+t\right)}{t \neq p \mathfrak{Q}_{p}\left(\alpha_{s}\right)-\alpha_{s}}}{\prod_{s \in \mathcal{C}_{\boldsymbol{\alpha}, r}}\left(\alpha_{s}+j p\right)_{r}} \underset{\prod_{s \in \mathcal{P}_{\boldsymbol{\beta}, r}}\binom{\prod_{t=0}^{r-1}\left(\beta_{s}+j p+t\right)}{t \neq p \mathfrak{D}_{p}\left(\beta_{s}\right)-\beta_{s}} \prod_{s \in \mathcal{C}_{\boldsymbol{\mathcal { R }}, r}}\left(\beta_{s}+j p\right)_{r}}{ }
$$

By Lemma 8.3 , we know that $\mathcal{Q}_{\boldsymbol{\alpha}, \boldsymbol{\beta}}(j p)=\mathcal{Q}_{\mathfrak{D}_{p}(\boldsymbol{\alpha}), \mathfrak{D}_{p}(\boldsymbol{\beta})}(j) \omega$, where $\omega \in \mathbb{Z}_{(p)}^{*}$ and $\omega \equiv$ $1 \bmod p$. Whence,

$$
\mathcal{Q}_{\boldsymbol{\alpha}, \boldsymbol{\beta}}(j p+r)=\mathcal{Q}_{\mathfrak{D}_{p}(\boldsymbol{\alpha}), \mathfrak{D}_{p}(\boldsymbol{\beta})}(j) \cdot \omega \cdot \lambda_{j} \cdot \xi
$$

We put $\nu=\omega \xi$. We now prove that $\nu \in \mathbb{Z}_{(p)}^{*}$ and that $\nu \bmod p=(\tau \bmod p)(\theta \bmod p)$. Since $r<p$, it is clear that

$$
\begin{equation*}
\prod_{\substack{t=0 \\ t \neq p \mathfrak{D}_{p}\left(\alpha_{s}\right)-\alpha_{s}}}^{r-1}\left(\alpha_{s}+j p+t\right) \bmod p \equiv \prod_{\substack{t=0 \\ t \neq p \mathfrak{D}_{p}\left(\alpha_{s}\right)-\alpha_{s}}}^{r-1}\left(\alpha_{s}+t\right) \bmod p \neq 0 \tag{8.9}
\end{equation*}
$$

and that

$$
\begin{equation*}
\prod_{\substack{t=0 \\ t \neq p \mathfrak{D}_{p}\left(\beta_{s}\right)-\beta_{s}}}^{r-1}\left(\beta_{s}+j p+t\right) \bmod p \equiv \prod_{\substack{t=0 \\ t \neq p \mathfrak{D}_{p}\left(\beta_{s}\right)-\beta_{s}}}^{r-1}\left(\beta_{s}+t\right) \bmod p \neq 0 \tag{8.10}
\end{equation*}
$$

So, it follows from Equations (8.9) and (8.10) that

$$
\prod_{s \in \mathcal{P}_{\boldsymbol{\alpha}, r}}\left(\prod_{\substack{t=0 \\ t \neq p \mathfrak{D}_{p}\left(\alpha_{s}\right)-\alpha_{s}}}^{r-1}\left(\alpha_{s}+j p+t\right)\right) / \prod_{s \in \mathcal{P}_{\mathcal{\beta}, r}}\left(\prod_{\substack{t=0 \\ t \neq p \mathfrak{D}_{p}\left(\beta_{s}\right)-\beta_{s}}}^{r-1}\left(\beta_{s}+j p+t\right)\right) \equiv \tau \bmod p
$$

Furthermore, it is not hard to see that, $s \in \mathcal{P}_{\alpha, r}$ if and only if $\left(\alpha_{s}+j p\right)_{r} \in p \mathbb{Z}_{(p)}$ and that, $s \in \mathcal{P}_{\boldsymbol{\beta}, r}$ if and only if $\left(\beta_{s}+j p\right)_{r} \in p \mathbb{Z}_{(p)}$. For this reason, the $p$-adic valuation of $\prod_{s \in \mathcal{C}_{\alpha, r}}\left(\alpha_{s}+j p\right)_{r}$ and $\prod_{s \in \mathcal{C}_{\boldsymbol{\beta}, r}}\left(\beta_{s}+j p\right)_{r}$ is zero. So

$$
\prod_{s \in \mathcal{C}_{\alpha, r}}\left(\alpha_{s}+j p\right)_{r} / \prod_{s \in \mathcal{C}_{\boldsymbol{\beta}, r}}\left(\beta_{s}+j p\right)_{r} \equiv \theta \bmod p
$$

Consequently, $\xi \in \mathbb{Z}_{(p)}^{*}$ and $\xi \bmod p=(\tau \bmod p)(\theta \bmod p)$. Finally, we know that $\omega \in$ $\mathbb{Z}_{(p)}^{*}$ and that $\omega \bmod p=1$. So, $\nu \in \mathbb{Z}_{(p)}^{*}$ and $\nu \bmod p=(\tau \bmod p)(\theta \bmod p)$.
(4). Let $j \geqslant 1$ be an integer. From (3), we have $\mathcal{Q}_{\boldsymbol{\alpha}, \boldsymbol{\beta}}(j p+r)=\mathcal{Q}_{\mathfrak{D}_{p}(\boldsymbol{\alpha}), \mathfrak{D}_{p}(\boldsymbol{\beta}), p}(j) \lambda_{j} \nu$, where $\nu \in \mathbb{Z}_{(p)}^{*}$. So $v_{p}\left(\mathcal{Q}_{\boldsymbol{\alpha}, \boldsymbol{\beta}}(j p+r)\right)=v_{p}\left(\mathcal{Q}_{\mathfrak{D}_{p}(\boldsymbol{\alpha}), \mathfrak{D}_{p}(\boldsymbol{\beta}), p}(j) \lambda_{j}\right)$. But, by assumption, we know that $v_{p}\left(\mathcal{Q}_{\boldsymbol{\alpha}, \boldsymbol{\beta}}(j p+r)\right) \geqslant 0$. Whence, $v_{p}\left(\mathcal{Q}_{\mathfrak{D}_{p}(\boldsymbol{\alpha}), \mathfrak{D}_{p}(\boldsymbol{\beta}), p}(j) \lambda_{j}\right) \geqslant 0$.

Now, it is clear that

$$
\begin{equation*}
\lambda_{0}\left(\frac{\prod_{s \in \mathcal{C}_{\alpha, r}} \mathfrak{D}_{p}\left(\alpha_{s}\right)_{j} \prod_{s \in \mathcal{P}_{\alpha, r}}\left(\mathfrak{D}_{p}\left(\alpha_{s}\right)+1\right)_{j}}{\prod_{s \in \mathcal{C}_{\boldsymbol{\beta}, r}} \mathfrak{D}_{p}\left(\beta_{s}\right)_{j} \prod_{s \in \mathcal{P}_{\boldsymbol{\beta}, r}}\left(\mathfrak{D}_{p}\left(\beta_{s}\right)+1\right)_{j}}\right)=\mathcal{Q}_{\mathfrak{D}_{p}(\boldsymbol{\alpha}), \mathfrak{D}_{p}(\boldsymbol{\beta})}(j) \lambda_{j} . \tag{8.11}
\end{equation*}
$$

By (1), we know that $v_{p}\left(\lambda_{0}\right)=0$. So,

$$
v_{p}\left(\frac{\prod_{s \in \mathcal{C}_{\boldsymbol{\alpha}, r}} \mathfrak{D}_{p}\left(\alpha_{s}\right)_{j} \prod_{s \in \mathcal{P}_{\boldsymbol{\alpha}, r}}\left(\mathfrak{D}_{p}\left(\alpha_{s}\right)+1\right)_{j}}{\prod_{s \in \mathcal{C}_{\boldsymbol{\beta}, r}} \mathfrak{D}_{p}\left(\beta_{s}\right)_{j} \prod_{s \in \mathcal{P}_{\boldsymbol{\beta}, r}}\left(\mathfrak{D}_{p}\left(\beta_{s}\right)+1\right)_{j}}\right)=v_{p}\left(\mathcal{Q}_{\mathfrak{D}_{p}(\boldsymbol{\alpha}), \mathfrak{D}_{p}(\boldsymbol{\beta}), p}(j) \lambda_{j}\right) \geqslant 0 .
$$

(5). Let $j \geqslant 1$ be an integer. From (3) and (4) we get

$$
\mathcal{Q}_{\boldsymbol{\alpha}, \boldsymbol{\beta}}(j p+r) \bmod p=\left(\mathcal{Q}_{\mathfrak{D}_{p}(\boldsymbol{\alpha}), \mathfrak{D}_{p}(\boldsymbol{\beta}), p}(j) \lambda_{j} \bmod p\right)(\tau \bmod p)(\theta \bmod p) .
$$

So, from Equation (8.11), we get

$$
\begin{aligned}
& \mathcal{Q}_{\boldsymbol{\alpha}, \boldsymbol{\beta}}(j p+r) \bmod p= \\
& \left(\frac{\prod_{s \in \mathcal{C}_{\boldsymbol{\alpha}, r}} \mathfrak{D}_{p}\left(\alpha_{s}\right)_{j} \prod_{s \in \mathcal{P}_{\boldsymbol{\alpha}, r}}\left(\mathfrak{D}_{p}\left(\alpha_{s}\right)+1\right)_{j}}{\prod_{\mathcal{\mathcal { \beta }}, r}} \mathfrak{D}_{p}\left(\beta_{s}\right)_{j} \prod_{s \in \mathcal{P}_{\boldsymbol{\beta}, r}}\left(\mathfrak{D}_{p}\left(\beta_{s}\right)+1\right)_{j}\right. \\
& \bmod p)\left(\lambda_{0} \bmod p\right)(\tau \bmod p)(\theta \bmod p) .
\end{aligned}
$$

From (2), we conclude that $\mathcal{Q}_{\boldsymbol{\alpha}, \boldsymbol{\beta}}(r) \bmod p=\left(\lambda_{0} \bmod p\right)(\tau \bmod p)(\theta \bmod p)$. Consequently,

$$
\mathcal{Q}_{\boldsymbol{\alpha}, \boldsymbol{\beta}}(j p+r) \bmod p=\left(\frac{\prod_{s \in \mathcal{C}_{\alpha, r}} \mathfrak{D}_{p}\left(\alpha_{s}\right)_{j} \prod_{s \in \mathcal{P}_{\boldsymbol{\alpha}, r}}\left(\mathfrak{D}_{p}\left(\alpha_{s}\right)+1\right)_{j}}{\prod_{s \in \mathcal{C}_{\boldsymbol{\beta}, r}} \mathfrak{D}_{p}\left(\beta_{s}\right)_{j} \prod_{s \in \mathcal{P}_{\boldsymbol{\beta}, r}}\left(\mathfrak{D}_{p}\left(\beta_{s}\right)+1\right)_{j}} \bmod p\right)\left(\mathcal{Q}_{\boldsymbol{\alpha}, \boldsymbol{\beta}}(r) \bmod p\right)
$$

### 8.5. Proof of Lemma 8.1

Let us write $E_{\boldsymbol{\alpha}, \boldsymbol{\beta}, p}=\left\{e_{0}, e_{1}, \ldots, e_{k}\right\}$, where $e_{i}<e_{i+1}$ for all $i \in\{0, \ldots, k\}$ and $e_{0}=0$. We set $e_{k+1}=p$ and we also write $S_{\boldsymbol{\alpha}, \boldsymbol{\beta}, p}=\left\{0, r_{1}, \ldots, r_{t}\right\}$. Recall that $f(z)$ is the power series $\sum_{j \geqslant 0} \mathcal{Q}_{\boldsymbol{\alpha}, \boldsymbol{\beta}}(j) z^{j} \in \mathbb{Z}_{(p)}[[z]]$. By hypotheses, we know that $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ belong to $\mathbb{Z}_{(p)}^{n}$. So, by Lemma 8.2, we have

$$
f(z)=P_{0}(z) \Lambda_{0}(f)^{p}+P_{r_{1}}(z) \Lambda_{r_{1}}(f)^{p}+\cdots+P_{r_{t}}(z) \Lambda_{r_{t}}(f)^{p} \bmod p
$$

where, for all $r_{i} \in S_{\boldsymbol{\alpha}, \boldsymbol{\beta}, p}, P_{r_{i}}(z)=\sum_{s=r_{i}}^{e_{j}-1}\left(\frac{\mathcal{Q}_{\alpha, \boldsymbol{\beta}}(s)}{\mathcal{Q}_{\boldsymbol{\alpha}, \boldsymbol{\beta}}\left(r_{i}\right)} \bmod p\right) z^{s}$ with $e_{j-1}=r_{i}$. By definition,

$$
\Lambda_{r_{i}}(f)=\sum_{j \geqslant 0} \mathcal{Q}_{\boldsymbol{\alpha}, \boldsymbol{\beta}}\left(j p+r_{i}\right) z^{j}
$$

Now, by assumption, we know that $\mathfrak{D}_{p}(\boldsymbol{\alpha}), \mathfrak{D}_{p}(\boldsymbol{\beta})$ belong to $\left(\mathbb{Z}_{(p)}^{*}\right)^{n}$. Further, for all integers $j \geqslant 0, v_{p}\left(\mathcal{Q}_{\boldsymbol{\alpha}, \boldsymbol{\beta}}(j)\right) \geqslant 0$ because $f(z) \in \mathbb{Z}_{(p)}[[z]]$ and, by definition, $v_{p}\left(\mathcal{Q}_{\boldsymbol{\alpha}, \boldsymbol{\beta}}\left(r_{i}\right)\right)=0$ for all $r_{i} \in S_{\boldsymbol{\alpha}, \boldsymbol{\beta}, p}$. Therefore, by (4) of Lemma 8.7, we conclude that, for all $r_{i} \in S_{\boldsymbol{\alpha}, \boldsymbol{\beta}, p}$,

$$
f_{1, r_{i}}(z)={ }_{n} F_{n-1}\left(\boldsymbol{\alpha}_{1, r_{i}}, \boldsymbol{\beta}_{1, r_{i}} ; z\right)=\sum_{m \geqslant 0}\left(\frac{\prod_{s \in \mathcal{C}_{\boldsymbol{\alpha}, r_{i}}} \mathfrak{D}_{p}\left(\alpha_{s}\right)_{m} \prod_{s \in \mathcal{P}_{\boldsymbol{\alpha}, r_{i}}}\left(\mathfrak{D}_{p}\left(\alpha_{s}\right)+1\right)_{m}}{\prod_{s \in \mathcal{C}_{\boldsymbol{\beta}, r_{i}}} \mathfrak{D}_{p}\left(\beta_{s}\right)_{m} \prod_{s \in \mathcal{P}_{\boldsymbol{\beta}, r_{i}}}\left(\mathfrak{D}_{p}\left(\beta_{s}\right)+1\right)_{m}}\right) z^{m}
$$

belongs to $1+z \mathbb{Z}_{(p)}[[z]]$. Furthermore, we deduce from (5) of Lemma 8.7 that, for all $i \in$ $\{0, \ldots, t\}$,

$$
\Lambda_{r_{i}}(f) \bmod p=\mathcal{Q}_{\boldsymbol{\alpha}, \boldsymbol{\beta}}\left(r_{i}\right) \bmod p \cdot f_{1, r_{i}}(z) \bmod p
$$

Therefore,

$$
f(z) \equiv Q_{0}(z) f_{1,0}^{p}+Q_{r_{1}} f_{1, r_{1}}(z)^{p}+\cdots+Q_{r_{t}}(z) f_{1, r_{t}}(z)^{p} \bmod p,
$$

where $Q_{r_{i}}(z)=\mathcal{Q}_{\boldsymbol{\alpha}, \boldsymbol{\beta}}\left(r_{i}\right) P_{r_{i}}(z)$.

## 9. Constructing the polynomial $P_{p}(Y)$

In this section we show how to obtain the polynomial $P_{p}(Y)$. Let $\boldsymbol{\alpha}=\left(\alpha_{1}, \ldots, \alpha_{n}\right)$, $\boldsymbol{\beta}=\left(\beta_{1}, \ldots, \beta_{n-1}, 1\right)$ be in $(\mathbb{Q} \cap(0,1])^{n}$ and let $p$ be a prime number such that $p>2 d_{\boldsymbol{\alpha}, \boldsymbol{\beta}}$ and $f(z):={ }_{n} F_{n-1}(\boldsymbol{\alpha}, \boldsymbol{\beta} ; z)$ belongs to $\mathbb{Z}_{(p)}[[z]]$ and let $l$ be the order of $p$ in $\left(\mathbb{Z} / d_{\boldsymbol{\alpha}, \boldsymbol{\beta}} \mathbb{Z}\right)^{*}$. As $p>2 d_{\boldsymbol{\alpha}, \boldsymbol{\beta}}$ then $\boldsymbol{\alpha}, \boldsymbol{\beta}$ belong to $\left(\mathbb{Z}_{(p)}^{*}\right)^{n}$ and, by Remark 3.4, $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ satisfies the $\mathbf{P}_{p, l}$ property. Then, by Proposition 4.1, for every $r \in S_{\mathfrak{D}_{p}^{l-1}(\boldsymbol{\alpha}), \mathfrak{D}_{p}^{l-1}(\boldsymbol{\beta}), p}, f_{l, r} \in 1+z \mathbb{Z}_{(p)}[[z]]$ and

$$
\begin{equation*}
f_{l, r} \equiv \sum_{j \in S_{\mathfrak{O}_{p}^{l-1}(\boldsymbol{\alpha}), \mathcal{D}_{p}^{l-1}(\boldsymbol{\beta}), p}} Q_{r, j}(z) f_{l, j}^{p^{l}} \bmod p \tag{9.1}
\end{equation*}
$$

where, for every $j \in S_{\mathfrak{D}_{p}^{l-1}(\boldsymbol{\alpha}), \mathfrak{D}_{p}^{l-1}(\boldsymbol{\beta}), p}, Q_{i, j}(z)$ belongs to $\mathbb{Z}_{(p)}[z]$ and has degree less than $p^{l}$. By following the proof of Theorem 3.2, the polynomial $P_{p}(Y)$ results from applying Proposition 4.2 to the system (9.1). Thus $P_{p}(Y)$ is obtained by subsequent elimination of the series $f_{l, j}^{p^{l}}$ for $j \in S_{\mathfrak{D}_{p}^{l-1}(\boldsymbol{\alpha}), \mathfrak{D}_{p}^{l-1}(\boldsymbol{\beta}), p} \backslash\{0\}^{(5)}$. It follows from the proof of Proposition 4.2 that this subsequent elimination is explicit once the polynomials $Q_{r, j}$ are known. Lemma 9.1 gives a formula for each polynomial $Q_{r, j}$. This formula is given recursively and is constructed from the polynomials given by the conclusion of Lemma 8.1. In order to state the lemma, we introduce the following polynomials. Let $r$ be in $S_{\mathfrak{D}_{p}^{l-1}(\boldsymbol{\alpha}), \mathfrak{D}_{p}^{l-1}(\boldsymbol{\beta}), p}$ and let us consider the vectors $\boldsymbol{\alpha}_{l, r}=\boldsymbol{\omega}=\left(\omega_{1}, \ldots, \omega_{n}\right)$ and $\boldsymbol{\beta}_{l, r}=\boldsymbol{\eta}=\left(\eta_{1}, \ldots, \eta_{n}\right) .^{(6)}$. For every $j \in S_{\boldsymbol{\omega}, \boldsymbol{\eta}, p}$,

[^5]we set
$$
T_{r, j}(z)=\sum_{s=j}^{j^{\prime}-1}\left(\mathcal{Q}_{\boldsymbol{\omega}, \boldsymbol{\eta}}(s) \bmod p\right) z^{s}
$$
where $j^{\prime}$ is defined as follows. If $j \neq \max E_{\boldsymbol{\omega}, \boldsymbol{\eta}, p}$ then $j^{\prime}$ is the element in $E_{\boldsymbol{\omega}, \boldsymbol{\eta}, p}$ such that $\left(j, j^{\prime}\right) \cap E_{\boldsymbol{\omega}, \boldsymbol{\eta}, p}=\emptyset$ or otherwise, $j^{\prime}=p$.

Let $k$ be in $\{2, \ldots, l\}$. For every $j \in S_{\mathfrak{D}_{p}^{k-1}(\boldsymbol{\omega}), \mathfrak{D}_{p}^{k-1}(\boldsymbol{\eta}), p}$ and $b \in S_{\mathfrak{D}_{p}^{k-2}(\boldsymbol{\omega}), \mathfrak{D}_{p}^{k-2}(\boldsymbol{\eta}), p}$, we set

$$
T_{r, j}^{(k-1, b)}=\sum_{s=\tau(j)}^{j^{\prime}-1}\left(\mathcal{Q}_{\boldsymbol{\omega}_{k-1, b}, \boldsymbol{\eta}_{k-1, b}}(s) \bmod p\right) z^{s},
$$

where $\tau: S_{\mathfrak{D}_{p}^{k-1}(\boldsymbol{\omega}), \mathfrak{D}_{p}^{k-1}(\boldsymbol{\eta}), p} \rightarrow S_{\boldsymbol{\omega}_{k-1, b}, \boldsymbol{\eta}_{k-1, b}, p}$ is the function given by Lemma 6.1 and $j^{\prime}$ is defined as follows. If $\tau(j) \neq \max E_{\boldsymbol{\omega}_{k-1, b}, \boldsymbol{\eta}_{k-1, b}, p}$ then $j^{\prime}$ is the element in $E_{\boldsymbol{\omega}_{k-1, b}, \boldsymbol{\eta}_{k-1, b}, p}$ such that $\left(\tau(j), j^{\prime}\right) \cap E_{\boldsymbol{\omega}_{k-1, b}, \boldsymbol{\eta}_{k-1, b}, p}=\emptyset$ or otherwise $j^{\prime}=p$.

We are now ready to state Lemma 9.1.
Lemma 9.1. - Let the assumptions be as in Proposition 4.1. If $l \geqslant 2$ then, for every $r, j \in S_{\mathfrak{D}_{p}^{l-1}(\boldsymbol{\alpha}), \mathfrak{D}_{p}^{l-1}(\boldsymbol{\beta}), p}$,

$$
\begin{aligned}
& Q_{r, j}= \\
& \left(\sum_{j_{l-1} \in S_{\mathfrak{O}_{p}^{l-2}(\boldsymbol{\omega}), \mathcal{D}_{p}^{l-2}(\eta), p}} \cdots \sum_{j_{1} \in S_{\omega, \eta, p}} T_{r, j_{1}}\left(T_{r, j_{2}}^{\left(1, j_{1}\right)}\right)^{p} \cdots\left(T_{r, j_{l-1}}^{\left(l-2, j_{l-2}\right)}\right)^{p^{l-2}}\right)\left(T_{r, j}^{\left(l-1, j_{l-1}\right)}\right)^{p^{l-1}} .
\end{aligned}
$$

If $l=1$ then, for every $r, j \in S_{\boldsymbol{\alpha}, \boldsymbol{\beta}, p}$,

$$
Q_{r, j}=T_{r, \tau(j)},
$$

where $\tau: S_{\boldsymbol{\alpha}, \boldsymbol{\beta}} \rightarrow S_{\boldsymbol{\omega}, \boldsymbol{\eta}, p}$ is the bijective map given by A) of Lemma 6.1.
Proof. - Let $r$ be in $S_{\mathfrak{D}_{p}^{l-1}(\boldsymbol{\alpha}), \mathfrak{D}_{p}^{l-1}(\boldsymbol{\beta}), p}$. Let $F$ be the hypergeometric series with parameters $\boldsymbol{\alpha}_{l, r}=\boldsymbol{\omega}=\left(\omega_{1}, \ldots, \omega_{n}\right)$ and $\boldsymbol{\beta}_{l, r}=\boldsymbol{\eta}=\left(\eta_{1}, \ldots, \eta_{n}\right)$. Then $F=f_{l, r}$. For every $0 \leqslant a<l$ and $j \in S_{\mathfrak{D}_{p}^{a}(\boldsymbol{\omega}), \mathfrak{D}_{p}^{a}(\boldsymbol{\eta}), p}$, we put $F_{a+1, j}={ }_{n} F_{n-1}\left(\boldsymbol{\omega}_{a+1, j}, \boldsymbol{\eta}_{a+1, j} ; z\right)$. That is,

$$
F_{a+1, j}=\sum_{m \geqslant 0}\left(\frac{\prod_{s \in \mathcal{C}_{\mathfrak{D}_{p}^{a}(\omega), j}} \mathfrak{D}_{p}^{a+1}\left(\omega_{s}\right)_{m} \prod_{s \in \mathcal{P}_{\mathfrak{D}_{p}^{a}(\omega), j}}\left(\mathfrak{D}_{p}^{a+1}\left(\omega_{s}\right)+1\right)_{m}}{\prod_{s \in \mathcal{C}_{\mathfrak{D}_{p}^{a}(\eta), j}} \mathfrak{D}_{p}^{a+1}\left(\eta_{s}\right)_{m} \prod_{s \in \mathcal{P}_{\mathfrak{D}_{p}^{a}(\eta), j}}\left(\mathfrak{D}_{p}^{a+1}\left(\eta_{s}\right)+1\right)_{m}}\right) z^{m} .
$$

As $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ satisfies the $\mathbf{P}_{p, l}$ property then, by (2) of Remark 6.5, $(\boldsymbol{\omega}, \boldsymbol{\eta})$ satisfies the $\mathbf{P}_{p, l}$ property and $l$ is the order of $p$ in $\left(\mathbb{Z} / d_{\boldsymbol{\omega}, \eta} \mathbb{Z}\right)^{*}$. Thus, by Lemma $6.2, F_{a+1, j} \in 1+z \mathbb{Z}_{(p)}[[z]]$ for all $0 \leqslant a<l$ and $j \in S_{\mathfrak{D}_{p}^{a}(\boldsymbol{\omega}), \mathfrak{D}_{p}^{a}(\boldsymbol{\eta}), p}$.

We first prove by induction on $k \in\{1, \ldots, l\}$ that

$$
\begin{equation*}
F=\sum_{j \in S_{\mathfrak{O}_{p}^{k-1}(\boldsymbol{\omega}), \mathcal{D}_{p}^{k-1}(\boldsymbol{\eta}), p}} Q_{r, j}^{(k-1)} F_{k, j}^{p^{k}}, \tag{9.2}
\end{equation*}
$$

where

$$
Q_{r, j}^{(k-1)}=\sum_{j_{k-1} \in S_{\mathcal{O}_{p}^{k-2}(\omega), \mathfrak{刃}_{p}^{k-2}(\eta), p}} \cdots \sum_{j_{1} \in S_{\boldsymbol{\omega}, \eta, p}} T_{r, j_{1}}\left(T_{r, j_{2}}^{\left(1, j_{1}\right)}\right)^{p} \cdots\left(T_{r, j}^{\left(k-1, j_{k-1}\right)}\right)^{p^{k-1}}
$$

For $k=1$, according to Lemma 8.1, we have

$$
F=\sum_{j \in S_{\omega, \eta, p}} T_{r, j}(z) F_{1, j}^{p}
$$

We now suppose that for some $k \in\{1, \ldots l-1\}$ Equality (9.2) holds. We are going to see that Equation 9.2 also holds for $k+1$. Let $j$ be in $S_{\mathfrak{D}_{p}^{k-1}(\boldsymbol{\omega}), \mathfrak{D}_{p}^{k-1}(\boldsymbol{\eta}), p}$. By definition, $F_{k, j}$ is the hypergeometric series ${ }_{n} F_{n-1}\left(\boldsymbol{\omega}_{k, j}, \boldsymbol{\eta}_{k, j} ; z\right)$. Further, we know that $(\boldsymbol{\omega}, \boldsymbol{\eta})$ satisfies the $\mathbf{P}_{p, l}$ property and thus, by Remark 6.4, $\left(\boldsymbol{\omega}_{k, j}, \boldsymbol{\eta}_{k, j}\right)$ satisfies the $\mathbf{P}_{p, l^{\prime}}$ property, where $l^{\prime}$ is the order $p$ in $\left(\mathbb{Z} / d_{\boldsymbol{\omega}_{k, j}, \boldsymbol{\eta}_{k, j}} \mathbb{Z}\right)^{*}$. So, by applying Lemma 8.1 to $F_{k, j}$, we get

$$
F_{k, j}=\sum_{\gamma \in S_{\omega_{k, j}, \eta_{k, j}}} Q_{\gamma} F_{\gamma}^{p}
$$

where

$$
F_{\gamma}=\sum_{m \geqslant 0}\left(\frac{\prod_{s \in \mathcal{C}_{\boldsymbol{\omega}_{k, j}, \gamma}} \mathfrak{D}_{p}\left(\omega_{s, k, j}\right)_{m} \prod_{s \in \mathcal{P}_{\boldsymbol{\omega}_{k, j}, \gamma}}\left(\mathfrak{D}_{p}\left(\omega_{s, k, j}\right)+1\right)_{m}}{\prod_{\boldsymbol{C}_{k, j}, \gamma}} \mathfrak{D}_{p}\left(\eta_{s, k, j}\right)_{m} \prod_{s \in \mathcal{P}_{\boldsymbol{\eta}_{k, j}, \gamma}}\left(\mathfrak{D}_{p}\left(\eta_{s, k, j}\right)+1\right)_{m}\right) z^{m}, Q_{\gamma}=\sum_{s=\gamma}^{\gamma^{\prime}-1} \mathcal{Q}_{\boldsymbol{\omega}_{k, j}, \boldsymbol{\eta}_{k, j}}(s) z^{s}
$$

and $\gamma^{\prime}$ is defined as follows. If $\gamma \neq \max E_{\boldsymbol{\omega}_{k, j}, \boldsymbol{\eta}_{k, j}}$ then $\gamma^{\prime}$ is the element in $E_{\boldsymbol{\omega}_{k, j}, \boldsymbol{\eta}_{k, j}}$ such that $\left(\gamma, \gamma^{\prime}\right) \cap E_{\boldsymbol{\omega}_{k, j}, \boldsymbol{\eta}_{k, j}}=\emptyset$ or otherwise, $\gamma^{\prime}=p$.

We now prove that $F_{\gamma}=F_{k+1, \sigma(\gamma)}$, where $\sigma: S_{\boldsymbol{\omega}_{k, j}, \boldsymbol{\eta}_{k, j}, p} \rightarrow S_{\mathfrak{D}_{p}^{k}(\boldsymbol{\omega}), \mathfrak{D}_{p}^{k}(\boldsymbol{\eta}), p}$ is the function given by Lemma $6.1^{(7)}$. For this purpose, we first prove that, for all $s \in\{1, \ldots, n\}$, $\mathfrak{D}_{p}\left(\omega_{s, k, j}\right)=\mathfrak{D}_{p}^{k+1}\left(\omega_{s}\right)$. By definition $\omega_{s, k, j}=\mathfrak{D}_{p}^{k}\left(\omega_{s}\right)$ if $s \in \mathcal{C}_{\mathfrak{D}_{p}^{k-1}(\boldsymbol{\omega}), j}$ or $\omega_{s, k, j}=\mathfrak{D}_{p}^{k}\left(\omega_{s}\right)+1$ if $s \in \mathcal{P}_{\mathfrak{D}_{p}^{k-1}(\boldsymbol{\omega}), j}$. It is clear that in the first case $\mathfrak{D}_{p}\left(\omega_{s, k, j}\right)=\mathfrak{D}_{p}^{k+1}\left(\omega_{s}\right)$. Suppose now that $\omega_{s, k, j}=\mathfrak{D}_{p}^{k}\left(\omega_{s}\right)+1$. Again, by definition $\omega_{s}=\alpha_{s, l, i}$ and thus $\omega_{s}=\alpha_{s}$ if $s \in \mathcal{C}_{\mathfrak{D}_{p}^{l-1}(\boldsymbol{\alpha}), i}$ or $\omega_{s}=\alpha_{s}+1$ if $s \in \mathcal{P}_{\mathfrak{D}_{p}^{l-1}(\boldsymbol{\alpha}), i}$. By assumption, $\alpha_{s} \in \mathbb{Z}_{(p)}^{*}$ and thus, by (1) of Remark 6.3, $\mathfrak{D}_{p}\left(\omega_{s}\right)=\mathfrak{D}_{p}\left(\alpha_{s}\right)$. In addition, for all integers $1 \leqslant r \leqslant l, \mathfrak{D}_{p}^{r}\left(\omega_{s}\right) \in \mathbb{Z}_{(p)}^{*}$ given that $\mathfrak{D}_{p}\left(\omega_{s}\right)=\mathfrak{D}_{p}\left(\alpha_{s}\right)$ and $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ satisfies the $\mathbf{P}_{p, l}$ property. Hence, according to (1) of Remark 6.3 again, $\mathfrak{D}_{p}\left(\omega_{s, k, j}\right)=\mathfrak{D}_{p}\left(\mathfrak{D}_{p}^{k}\left(\omega_{s}\right)+1\right)=\mathfrak{D}_{p}^{k+1}\left(\omega_{s}\right)$. In a similar way we show that, for all $s \in\{1, \ldots, n\}, \mathfrak{D}_{p}\left(\eta_{s, k, j}\right)=\mathfrak{D}_{p}^{k+1}\left(\eta_{s}\right)$. Now, by (ii) of Lemma 6.1, we have $\mathcal{C}_{\boldsymbol{\omega}_{k, j}, \gamma}=\mathcal{C}_{\mathfrak{D}_{p}^{k}(\boldsymbol{\omega}), \sigma(\gamma)}$ and $\mathcal{P}_{\boldsymbol{\omega}_{k, j}, \gamma}=\mathcal{P}_{\mathfrak{O}_{p}^{k}(\boldsymbol{\omega}), \sigma(\gamma)}$. Again, by (ii) of Lemma 6.1, we have $\mathcal{C}_{\boldsymbol{\eta}_{k, j}, \gamma}=\mathcal{C}_{\mathfrak{D}_{p}^{k}(\boldsymbol{\eta}), \sigma(\gamma)}$ and $\mathcal{P}_{\boldsymbol{\eta}_{k, j}, \gamma}=\mathcal{P}_{\mathfrak{D}_{p}^{k}(\boldsymbol{\eta}), \sigma(\gamma)}$. Consequently, $F_{\gamma}=F_{k+1, \sigma(\gamma)}$. Finally, it is clear that $Q_{\gamma}=T_{r, \sigma(\gamma)}^{(k, j)}$ because $\tau(\sigma(\gamma)=\gamma$. Therefore,

$$
F_{k, j}=\sum_{i \in S_{\mathfrak{O}_{p}^{k}(\boldsymbol{\omega}), \mathcal{D}_{p}^{k}(\eta), p}} T_{r, i}^{(k, j)} F_{k+1, i}^{p} .
$$

[^6]Thus, from induction hypothesis and the previous equality, we obtain

$$
\begin{aligned}
& F= \\
& \sum_{j \in S_{\mathfrak{O}_{p}^{k}(\omega), \mathcal{D}_{p}^{k}(\eta), p}}\left(\sum_{j_{k} \in S_{\mathcal{O}_{p}^{k-1}(\omega), \mathcal{D}_{p}^{k-1}(\eta), p}} \ldots \sum_{j_{1} \in S_{\omega, \eta, p}} T_{r, j_{1}}\left(T_{r, j_{2}}^{\left(1, j_{1}\right)}\right)^{p} \cdots\left(T_{r, j}^{\left(k, j_{k}\right)}\right)^{p^{k}}\right) F_{k+1, j}^{p^{k+1}},
\end{aligned}
$$

which shows that Equation 9.2 is true for $k+1$.
So, by induction we conclude that Equation (9.2) holds for all $1 \leqslant k \leqslant l$.
Suppose that $l \geqslant 2$. We will show that, for every $j \in S_{\mathfrak{D}_{p}^{l-1}(\boldsymbol{\omega}), \mathfrak{D}_{p}^{l-1}(\boldsymbol{\eta}), p}, F_{l, j}=f_{l, j}$. By (1) of Remark 6.5, we have $\mathfrak{D}_{p}^{l-1}(\boldsymbol{\omega})=\mathfrak{D}_{p}^{l-1}(\boldsymbol{\alpha})$ and $\mathfrak{D}_{p}^{l-1}(\boldsymbol{\eta})=\mathfrak{D}_{p}^{l-1}(\boldsymbol{\beta})$. Thus, $\mathcal{C}_{\mathfrak{D}_{p}^{l-1}(\boldsymbol{\omega}), j}=$ $\mathcal{C}_{\mathfrak{D}_{p}^{l-1}(\boldsymbol{\alpha}), j}, \mathcal{P}_{\mathfrak{D}_{p}^{l-1}(\boldsymbol{\omega}), j}=\mathcal{P}_{\mathfrak{D}_{p}^{l-1}(\boldsymbol{\alpha}), j}, \mathcal{C}_{\mathfrak{D}_{p}^{l-1}(\boldsymbol{\eta}), j}=\mathcal{C}_{\mathfrak{D}_{p}^{l-1}(\boldsymbol{\beta}), j}$, and $\mathcal{P}_{\mathfrak{D}_{p}^{l-1}(\boldsymbol{\eta}), j}=\mathcal{P}_{\mathfrak{D}_{p}^{l-1}(\boldsymbol{\beta}), j}$. Furthermore, we also have $S_{\mathfrak{D}_{p}^{l-1}(\boldsymbol{\omega}), \mathfrak{D}_{p}^{l-1}(\boldsymbol{\eta}), p}=S_{\mathfrak{D}_{p}^{l-1}(\boldsymbol{\alpha}), \mathfrak{D}_{p}^{l-1}(\boldsymbol{\beta}), p}$. By (1) of Remark 6.5 again, we have $\mathfrak{D}_{p}^{l}(\boldsymbol{\omega})=\mathfrak{D}_{p}^{l}(\boldsymbol{\alpha})$ and $\mathfrak{D}_{p}^{l}(\boldsymbol{\eta})=\mathfrak{D}_{p}^{l}(\boldsymbol{\beta})$. As $p^{l} \equiv 1 \bmod d_{\boldsymbol{\alpha}, \boldsymbol{\beta}}$ and, by assumption, $\boldsymbol{\alpha}$, $\boldsymbol{\beta} \in\left(\mathbb{Z}_{(p)}^{*}\right)^{n}$ then Lemma 2.1 implies that $\mathfrak{D}_{p}^{l}(\boldsymbol{\alpha})=\boldsymbol{\alpha}$ and $\mathfrak{D}_{p}^{l}(\boldsymbol{\beta})=\boldsymbol{\beta}$. So, $\mathfrak{D}_{p}^{l}(\boldsymbol{\omega})=\boldsymbol{\alpha}$ and $\mathfrak{D}_{p}^{l}(\boldsymbol{\eta})=\boldsymbol{\beta}$. Therefore, for every $j \in S_{\mathfrak{D}_{p}^{l-1}(\boldsymbol{\omega}), \mathfrak{D}_{p}^{l-1}(\boldsymbol{\eta}), p}, F_{l, j}=f_{l, j}$. Consequently, it follows from Equation (9.2) that

$$
F=\sum_{j \in S_{\mathfrak{D}_{p}^{l-1}(\alpha), \mathfrak{D}_{p}^{l-1}(\mathcal{\beta}), p}} Q_{r, j}(z) f_{l, j}^{p^{l}} \bmod p,
$$

where

$$
\begin{aligned}
& Q_{r, j}= \\
& \left(\sum_{j_{l-1} \in S_{\mathcal{O}_{p}^{l-2}(\boldsymbol{\omega}), \mathcal{D}_{p}^{l-2}(\eta), p}} \cdots \sum_{j_{1} \in S_{\boldsymbol{\omega}, \boldsymbol{\eta}, p}} T_{r, j_{1}}\left(T_{r, j_{2}}^{\left(1, j_{1}\right)}\right)^{p} \cdots\left(T_{r, j_{l-1}}^{\left(l-2, j_{l-2}\right)}\right)^{p^{l-2}}\right)\left(T_{r, j}^{\left(l-1, j_{l-1}\right)}\right)^{p^{l-1}} .
\end{aligned}
$$

This completes the case $l \geqslant 2$ because $F=f_{l, r}$.
Suppose now that $l=1$. As $p \equiv 1 \bmod d_{\boldsymbol{\alpha}, \boldsymbol{\beta}}$ and, by assumption, $\boldsymbol{\alpha}, \boldsymbol{\beta} \in\left(\mathbb{Z}_{(p)}^{*}\right)^{n}$ then Lemma 2.1 implies that, $\mathfrak{D}_{p}(\boldsymbol{\alpha})=\boldsymbol{\alpha}$ and $\mathfrak{D}_{p}(\boldsymbol{\beta})=\boldsymbol{\beta}$. By assumption again, $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ satisfies the $\mathbf{P}_{p, 1}$ property. Then, by Lemma 6.1, we have $\sigma: S_{\boldsymbol{\alpha}_{1, r}, \boldsymbol{\beta}_{1, l}, p} \rightarrow S_{\boldsymbol{\alpha}, \boldsymbol{\beta}, p}$. But definition, $\boldsymbol{\alpha}_{1, r}=\boldsymbol{\omega}$ and $\boldsymbol{\beta}_{1, r}=\boldsymbol{\eta}$. So, $\sigma: S_{\boldsymbol{\omega}, \boldsymbol{\eta}, p} \rightarrow S_{\boldsymbol{\alpha}, \boldsymbol{\beta}, p}$. We are going to see that, for all $j \in S_{\boldsymbol{\omega}, \boldsymbol{\eta}, p}$, $F_{1, j}=f_{1, \sigma(j)}$. By B) of Lemma 6.1, we get $\mathcal{C}_{\boldsymbol{\omega}, j}=\mathcal{C}_{\boldsymbol{\alpha}, \sigma(j)}, \mathcal{P}_{\boldsymbol{\omega}, j}=\mathcal{P}_{\boldsymbol{\alpha}, \sigma(j)}, \mathcal{C}_{\boldsymbol{\eta}, j}=\mathcal{C}_{\boldsymbol{\beta}, \sigma(j)}$, and $\mathcal{P}_{\boldsymbol{\eta}, j}=\mathcal{C}_{\boldsymbol{\alpha}, \sigma(j)}$. By (1) of Remark 6.5, we have $\mathfrak{D}_{p}(\boldsymbol{\omega})=\mathfrak{D}_{p}(\boldsymbol{\alpha})$ and $\mathfrak{D}_{p}(\boldsymbol{\eta})=\mathfrak{D}_{p}(\boldsymbol{\beta})$. But we know that $\mathfrak{D}_{p}(\boldsymbol{\alpha})=\boldsymbol{\alpha}$ and $\mathfrak{D}_{p}(\boldsymbol{\beta})=\boldsymbol{\beta}$. So $\mathfrak{D}_{p}(\boldsymbol{\omega})=\boldsymbol{\alpha}$ and $\mathfrak{D}_{p}(\boldsymbol{\eta})=\boldsymbol{\beta}$. Consequently, for all $j \in S_{\boldsymbol{\omega}, \boldsymbol{\eta}, p}, F_{1, j}=f_{1, \sigma(j)}$. Since $\sigma$ is a bijective map, we deduce from Equation (9.2) that

$$
F=\sum_{j \in S_{\alpha, \boldsymbol{\beta}, p}} Q_{r, j}(z) f_{1, j}^{p}
$$

where $Q_{r, j}=T_{r, \tau(j)}$ with $\tau$ the inverse of $\sigma$. This completes the case $l=1$ because $F=$ $f_{1, r}$.

As an application of Lemma 9.1, we will give a formula for each rational function appearing in Equations (2.1) and (2.2).

Theorem 9.2.- Let $\boldsymbol{\alpha}=\left(\alpha_{1}, \ldots, \alpha_{n}\right)$, $\boldsymbol{\beta}=\left(\beta_{1}, \ldots, \beta_{n-1}, 1\right)$ be in $(\mathbb{Q} \cap(0,1])^{n}$ and let $p$ be a prime number such that $p>2 d_{\boldsymbol{\alpha}, \boldsymbol{\beta}}$ and $f(z):={ }_{n} F_{n-1}(\boldsymbol{\alpha}, \boldsymbol{\beta} ; z)$ belongs to $\mathbb{Z}_{(p)}[[z]]$. Suppose that $\# S_{\boldsymbol{\alpha}, \boldsymbol{\beta}, p}=2$. We write $S_{\boldsymbol{\alpha}, \boldsymbol{\beta}, p}=\{0, r\}, E_{\boldsymbol{\alpha}, \boldsymbol{\beta}, p}=\left\{e_{0}, e_{1}, \ldots, e_{k}\right\}$ with $e_{0}=0$ and $e_{i}<e_{i+1}$ for all $i \in\{0, \ldots, k\}, E_{\boldsymbol{\alpha}_{1, r}, \boldsymbol{\beta}_{1, r}, p}=\left\{e_{0}^{\prime}, e_{1}^{\prime}, \ldots, e_{m}^{\prime}\right\}$ with $e_{0}^{\prime}=0$ and $e_{i}^{\prime}<e_{i+1}^{\prime}$ for all $i \in\{0, \ldots, m\}$. Let $e_{k+1}$ and $e_{m+1}^{\prime}$ be the prime number $p$. We put $r=e_{s-1}$ and $\tau(r)=e_{h-1}^{\prime}$. If $p \equiv 1 \bmod d_{\boldsymbol{\alpha}, \boldsymbol{\beta}}$ then

$$
f(z) \equiv Q_{1}(z) f(z)^{p}+Q_{2}(z) f^{p^{2}} \bmod p
$$

where

$$
Q_{1}(z)=P_{0}+\frac{T_{1}^{p}}{P_{1}^{p-1}} \text { and } Q_{2}(z)=P_{1} T_{0}^{p}-\frac{T_{1}^{p} P_{0}^{p}}{P_{1}^{p-1}}
$$

with

$$
P_{0}(z)=\sum_{j=0}^{e_{1}-1} \mathcal{Q}_{\boldsymbol{\alpha}, \boldsymbol{\beta}}(j) z^{j}, \quad P_{1}(z)=\sum_{j=r_{1}}^{e_{s}-1} \mathcal{Q}_{\boldsymbol{\alpha}, \boldsymbol{\beta}}(j) z^{j}
$$

and

$$
T_{0}(z)=\sum_{j=0}^{e_{1}^{\prime}-1} \mathcal{Q}_{\boldsymbol{\alpha}_{1, r}, \boldsymbol{\beta}_{1, r}}(j) z^{j}, \quad T_{1}(z)=\sum_{j=\tau\left(r_{1}\right)}^{e_{h}^{\prime}-1} \mathcal{Q}_{\boldsymbol{\alpha}_{1, r}, \boldsymbol{\beta}_{1, r}}(j) z^{j} z^{j}
$$

Proof. - It is clear that 1 is the order of $p$ in $\left(\mathbb{Z} / d_{\boldsymbol{\alpha}, \boldsymbol{\beta}} \mathbb{Z}\right)^{*}$. It follows from Remark 3.4, that $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ satisfies de $\mathbf{P}_{p, 1}$ property because $p>2 d_{\boldsymbol{\alpha}, \boldsymbol{\beta}}$. Further, $\boldsymbol{\alpha}, \boldsymbol{\beta}$ belong to $\left(\mathbb{Z}_{(p)}^{*}\right)^{n}$ because $p>2 d_{\boldsymbol{\alpha}, \boldsymbol{\beta}}$ and $\boldsymbol{\alpha}, \boldsymbol{\beta}$ belong to $(0,1]^{n}$. Note that $f$ is the hypergeometric series $f_{1,0}$ because Lemma 2.1 implies $\mathfrak{D}_{p}(\boldsymbol{\alpha})=\boldsymbol{\alpha}$ and $\mathfrak{D}_{p}(\boldsymbol{\beta})=\boldsymbol{\beta}$. Then, by Lemma 9.1, we get

$$
\begin{gather*}
f(z) \equiv P_{0}(z) f(z)^{p}+P_{1}(z) f_{1,1}(z)^{p} \bmod p  \tag{9.3}\\
f_{1,1}(z) \equiv T_{0}(z) f(z)^{p}+T_{1}(z) f_{1,1}(z)^{p} \bmod p \tag{9.4}
\end{gather*}
$$

where

$$
f_{1,1}(z)=\sum_{m \geqslant 0}\left(\frac{\prod_{s \in \mathcal{C}_{\alpha, r}} \mathfrak{D}_{p}\left(\alpha_{s}\right)_{m} \prod_{s \in \mathcal{P}_{\alpha, r}}\left(\mathfrak{D}_{p}\left(\alpha_{s}\right)+1\right)_{m}}{\prod_{s \in \mathcal{C}_{\boldsymbol{\beta}, r}} \mathfrak{D}_{p}\left(\beta_{s}\right)_{m} \prod_{s \in \mathcal{P}_{\boldsymbol{\beta}, r}}\left(\mathfrak{D}_{p}\left(\beta_{s}\right)+1\right)_{m}}\right) z^{m} \in 1+z \mathbb{Z}_{(p)}[[z]] .
$$

From Equations (9.3) and (9.4), we get

$$
f=\left(P_{0}+\frac{T_{1}^{p}}{P_{1}^{p-1}}\right) f^{p}+\left(P_{1} T_{0}^{p}-\frac{T_{1}^{p} P_{0}^{p}}{P_{1}^{p-1}}\right) f^{p^{2}} \bmod p
$$

As a corollary of Theorem 9.2 we have
Corollary 9.3. - Let $\mathfrak{f}(z):={ }_{2} F_{1}(\boldsymbol{\alpha}, \boldsymbol{\beta} ; z)$ with $\boldsymbol{\alpha}=\left(\frac{1}{3}, \frac{1}{2}\right)$ and $\boldsymbol{\beta}=\left(\frac{5}{12}, 1\right)$, and let $p$ be a prime number such that $p=1+12 k$ and $p>24$. Then, $\mathfrak{f}(z) \in \mathbb{Z}_{(p)}[[z]]$ and

$$
\mathfrak{f} \equiv\left(P_{0}+\frac{T_{1}^{p}}{P_{1}^{p-1}}\right) \mathfrak{f}^{p}+\left(P_{1} T_{0}^{p}-\frac{T_{1}^{p} P_{0}^{p}}{P_{1}^{p-1}}\right) \mathfrak{f}^{p^{2}} \bmod p
$$

where

$$
P_{0}(z)=\sum_{j=0}^{5 k} \frac{(1 / 3)_{j}(1 / 2)_{j}}{(5 / 12)_{j}(1)_{j}} z^{j}, P_{1}=\sum_{j=1+5 k}^{p-1} \frac{(1 / 3)_{j}(1 / 2)_{j}}{(5 / 12)_{j}(1)_{j}} z^{j}
$$

and

$$
T_{0}(z)=\sum_{j=0}^{5 k-1} \frac{(1 / 3+1)_{j}(1 / 2)_{j}}{(5 / 12+1)_{j}(1)_{j}} z^{j}, T_{1}(z)=\sum_{j=5 k}^{p-1} \frac{(1 / 3+1)_{j}(1 / 2)_{j}}{(5 / 12+1)_{j}(1)_{j}} z^{j}
$$

Proof. - Note that $E_{\boldsymbol{\alpha}, \boldsymbol{\beta}, p}=\{0,1+5 k\}$ and we have proved in Example 2.3 that $S_{\boldsymbol{\alpha}, \boldsymbol{\beta}, p}=$ $\{0,1+5 k\}$. From the calculations made in Example 2.3 it follows that $\boldsymbol{\alpha}_{1,1+5 k}=(1 / 3+1,1)$ and $\boldsymbol{\beta}_{1,1+5 k}=(5 / 12+1,1)$. Thus $E_{\boldsymbol{\alpha}_{1,1+5 k}, \boldsymbol{\beta}_{1,1+5 k}}=\{0,5 k\}$ and $\tau(1+5 k)=5 k$, where $\tau: S_{\mathfrak{D}_{p}(\boldsymbol{\alpha}), \mathfrak{D}_{p}(\boldsymbol{\beta}), p} \rightarrow S_{\boldsymbol{\alpha}_{1,1+5 k}, \boldsymbol{\beta}_{1,1+5 k}, p}$ is the function given by Lemma 6.1. Since $\mathfrak{D}_{p}(\boldsymbol{\alpha})=\boldsymbol{\alpha}$ and $\mathfrak{D}_{p}(\boldsymbol{\beta})=\boldsymbol{\beta}, S_{\mathfrak{D}_{p}(\boldsymbol{\alpha}), \mathfrak{D}_{p}(\boldsymbol{\beta}), p}=\{0,1+5 k\}$. Thus, from Theorem 9.2, we get

$$
\mathfrak{f} \equiv\left(P_{0}+\frac{T_{1}^{p}}{P_{1}^{p-1}}\right) \mathfrak{f}^{p}+\left(P_{1} T_{0}^{p}-\frac{T_{1}^{p} P_{0}^{p}}{P_{1}^{p-1}}\right) \mathfrak{f}^{p^{2}} \bmod p
$$

In the next theorem we give an explicit formula for each rational function appearing in Equation (2.2).

Theorem 9.4. - Let $\mathfrak{g}(z)={ }_{3} F_{2}(\boldsymbol{\alpha}, \boldsymbol{\beta} ; z)$ with $\boldsymbol{\alpha}=\left(\frac{1}{9}, \frac{4}{9}, \frac{5}{9}\right)$ and $\boldsymbol{\beta}=\left(\frac{1}{3}, 1,1\right)$, and let $p$ be a prime number such that $p=8+9 k_{p}$ and $p>18$. Then, $\mathfrak{g}(z) \in \mathbb{Z}_{(p)}[[z]]$ and

$$
\mathfrak{g} \equiv\left(P_{0,0} P_{1,0}^{p}+P_{0,0} P_{1,1}^{p}\left(\frac{R_{1,0}}{P_{0,0}}\right)^{p^{2}}\right) \mathfrak{g}^{p^{2}} \bmod p
$$

where

$$
P_{0,0}(z)=\sum_{s=0}^{5+6 k_{p}} \frac{(1 / 9)_{s}(4 / 9)_{s}(5 / 9)_{s}}{(1 / 3)_{s}(1)_{s}^{2}} z^{s}, P_{1,0}=\sum_{s=0}^{2+3 k_{p}} \frac{(8 / 9)_{s}(5 / 9)_{s}(4 / 9)_{s}}{(2 / 3)_{s}(1)_{s}^{2}} z^{s}
$$

and

$$
P_{1,1}=\sum_{s=3+3 k_{p}}^{p-1} \frac{(8 / 9)_{s}(5 / 9)_{s}(4 / 9)_{s}}{(2 / 3)_{s}(1)_{s}^{2}} z^{s}, R_{1,0}(z)=\sum_{s=0}^{4+6 k_{p}} \frac{(1 / 9+1)_{s}(4 / 9)_{s}(5 / 9)_{s}}{(1 / 3+1)_{s}(1)_{s}^{2}} z^{s}
$$

Proof. - From Example 2.4 we know that $E_{\mathfrak{D}_{p}(\boldsymbol{\alpha}), \mathfrak{D}_{p}(\boldsymbol{\beta})}=S_{\mathfrak{D}_{p}(\boldsymbol{\alpha}), \mathfrak{D}_{p}(\boldsymbol{\beta}), p}=\{0,3+3 k\}$. Furthermore, it is clear that 2 is the order of $p$ in $(\mathbb{Z} / 9 \mathbb{Z})^{*}$ and, according to Remark 3.4, $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ satisfies the $\mathbf{P}_{p, 2}$ property because $p>18$. Hence, by Lemma 9.1, we get

$$
\begin{gather*}
\mathfrak{g}(z) \equiv P_{0,0} P_{1,0}^{p} \mathfrak{g}^{p^{2}}+P_{0,0} P_{1,1}^{p} \mathfrak{g}_{1,1}^{p^{2}} \bmod p  \tag{9.5}\\
\mathfrak{g}_{1,1}(z) \equiv R_{1,0} P_{1,0}^{p} \mathfrak{g}^{p^{2}}+R_{1,0} P_{1,1}^{p} \mathfrak{g}_{1,1}^{p^{2}} \bmod p \tag{9.6}
\end{gather*}
$$

where $\mathfrak{g}_{1,1}$ is the hypergeometric series ${ }_{3} F_{2}(\boldsymbol{\omega}, \boldsymbol{\eta} ; z)$ with $\boldsymbol{\omega}=((1 / 9)+1,4 / 9,5 / 9)$ and $\boldsymbol{\eta}=((1 / 3)+1,1,1)$. Multiplying Equation (9.5) by $R_{1,0}$ and Equation (9.6) by $P_{0,0}$ and subtracting the equations obtained we deduce that

$$
R_{1,0} \mathfrak{g} \equiv P_{0,0} \mathfrak{g}_{1,1} \bmod p
$$

So $\mathfrak{g}_{1,1} \equiv \frac{R_{1,0}}{P_{0,0}} \mathfrak{g} \bmod p$. By replacing this last equality into (9.5) we obtain

$$
\mathfrak{g} \equiv\left(P_{0,0} P_{1,0}^{p}+P_{0,0} P_{1,1}^{p}\left(\frac{R_{1,0}}{P_{0,0}}\right)^{p^{2}}\right) \mathfrak{g}^{p^{2}} \bmod p
$$

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[^1]:    ${ }^{(1)}$ The existence of strong Frobenius structure of $\mathcal{H}(\boldsymbol{\alpha}, \boldsymbol{\beta})$ is a directly consequence of a result due to Crew [7]. The approach used by Crew is via $p$-adic cohomology. Nevertheless, we also obtain this result in [11, Theorem 6.2] by using an elementary approach based on ideas of Christol [5] and Saliner [10].

[^2]:    ${ }^{(2)}$ Let $\alpha$ be in $\mathbb{Z}_{(p)}$. Then, from the definition of $\mathfrak{D}_{p}$, it follows that the denominator of $\mathfrak{D}_{p}(\alpha)$ is a factor of the denominator of $\alpha$.

[^3]:    ${ }^{(3)}$ We refer the reader to [3, Section 2], where the authors explain why these operators are referred to as the Cartiers Operators.

[^4]:    ${ }^{(4)}$ If $J_{k,>1} \neq \emptyset$ then there is $i \in\{1, \ldots, n\}$ such that $v_{p}\left(\mathfrak{D}_{p}^{k}\left(\beta_{i}\right)_{j_{k}}\right)>1$. Since $j_{k}<p$, Step II implies $v_{p}\left(\mathfrak{D}_{p}^{k}\left(\beta_{i}\right)_{j_{k}}\right)=1$ which is a contradiction. Thus, $J_{k,>1}=\emptyset$.

[^5]:    ${ }^{(5)}$ Remember that, for every integer $a \geqslant 1,0 \in S_{\mathfrak{D}_{p}^{a-1}(\boldsymbol{\alpha}), \mathfrak{D}_{p}^{a-1}(\boldsymbol{\beta}), p}$. Note that $f_{l, 0}=f$.
    ${ }^{(6)}$ For every integer $a \geqslant 1$ and $r \in\{0, \ldots, p-1\}$, the definition of the vectors $\boldsymbol{\alpha}_{a, r}, \boldsymbol{\beta}_{a, r}$ was given at the beginning of Section 4.

[^6]:    ${ }^{(7)}$ Note that we can apply Lemma 6.1 to $(\boldsymbol{\omega}, \boldsymbol{\eta})$ because $(\boldsymbol{\omega}, \boldsymbol{\eta})$ satisfies the $\mathbf{P}_{p, l}$ property.

